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Progress in perturbative calculations for the anomalous magnetic moment of the muon

P. Marquard
Deutsches Elektronen-Synchrotron, DESY, Platanenallee 6, D-15738 Zeuthen, Germany

Abstract

We present a review on recent progress in perturbative calculations for the anomalous magnetic moment of the muon. We present recent calculations for leptonic contributions to $g-2$ and discuss the NNLO contributions to hadronic vacuum polarisation insertions.

Keywords: QED, $g-2$, hadronic contributions

1. Introduction

The anomalous magnetic moment of the muon $(g-2)_\mu$ has been both experimentally measured and theoretically calculated with astonishing precision. The difference between the experimental value [1, 2]

$$a_\mu^{\text{exp}} = 0.001\,165\,920\,80(54)(33)[63] \tag{1}$$

and the theory prediction [3]

$$a_\mu^{\text{theo}} = 0.001\,165\,918\,40(59) \tag{2}$$

has the size of about three standard deviations. On the theory side the contributions to a_μ^{theo} can be decomposed into three parts

$$a_\mu^{\text{theo}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{hadr}}\,, \tag{3}$$

where a_μ^{QED}, a_μ^{EW}, and a_μ^{hadr} denote the QED, electroweak, and hadronic contributions, respectively. The error on the theory prediction (2) is dominated by the uncertainty of the hadronic contributions [4, 5].

The QED contributions have been calculated at three loops analytically in [6] and numerically up to five-loop order in [3, 7]. We want to stress that looking at the absolute size of the QED corrections one finds that the four-loop contribution is of the same size as the difference between theory and experiment. Therefore it is mandatory to verify the only existing calculation of these contributions by an independent one. First steps towards this are presented in Section 2. Corrections to a_μ^{QED} from vacuum polarization insertions have been calculated up to five-loop order and are discussed in Section 4. Even though the corrections contained in a_μ^{hadr} are of non-perturbative nature they still receive quantum corrections which can be addressed in perturbation theory and are discussed in Section 3.

2. Leptonic contributions at four-loop order

The pure QED contributions can be further decomposed as

$$a_\mu^{\text{QED}} = \sum_{n=1} \left(\frac{\alpha}{\pi}\right)^n A_\mu^{(n)}\,, \tag{4}$$

$$\begin{aligned} A_\mu^{(n)} &= A_1^{(n)} + A_2^{(n)}(M_e/M_\mu) + A_2^{(n)}(M_\mu/M_\tau) \\ &\quad + A_3^{(n)}(M_e/M_\mu, M_\mu/M_\tau)\,, \end{aligned} \tag{5}$$

$$\begin{aligned} A_2^{(4)}(M_e/M_\mu) &= n_e^3 A_2^{(43)} + n_e^2 A_2^{(42)a} + n_e^2 n_\mu A_2^{(42)b} \\ &\quad + \cdots\,, \end{aligned} \tag{6}$$

Email address: peter.marquard@desy.de (P. Marquard)

where $A_1^{(n)}$ contains the universal mass-independent contribution. n_e and n_μ denote contributions from electron and muon loops, respectively.

In Ref. [8] a first step towards an independent calculation of the electronic contributions $A_2^{(4)}(M_e/M_\mu)$ to the anomalous moment of the muon has been made. Contributions with at least two closed electron loops have been calculated. The results are accurate up to terms M_e/M_μ and we show here only the new result for $A_2^{(42)a}$

$$A_2^{(42)a} = L_{\mu e}^2\left[\pi^2\left(\frac{5}{36}-\frac{a_1}{6}\right)+\frac{\zeta_3}{4}-\frac{13}{24}\right]+L_{\mu e}\left[-\frac{a_1^4}{9}+\pi^2\left(-\frac{2a_1^2}{9}+\frac{5a_1}{3}-\frac{79}{54}\right)-\frac{8a_4}{3}-3\zeta_3+\frac{11\pi^4}{216}+\frac{23}{6}\right] -\frac{2a_1^5}{45}+\frac{5a_1^4}{9}+\pi^2\left(-\frac{4a_1^3}{27}+\frac{10a_1^2}{9}-\frac{235a_1}{54}-\frac{\zeta_3}{8}+\frac{595}{162}\right)+\pi^4\left(-\frac{31a_1}{540}-\frac{403}{3240}\right)+\frac{40a_4}{3}+\frac{16a_5}{3} -\frac{37\zeta_5}{6}+\frac{11167\zeta_3}{1152}-\frac{6833}{864}\approx -3.624\,27\,, \quad (7)$$

with $L_{\mu e} = \ln(M_\mu^2/M_e^2)$, $\zeta_n = \sum_{k=1} 1/k^n$, $a_1 = \ln 2$ and $a_n = \mathrm{Li}_n(1/2)$, $n \geq 4$. Excellent agreement with the results in the literature [3, 9–11] has been found.

The contributions from τ-leptons to the anomalous magnetic moment of the muon can very efficiently be calculated by performing an asymptotic expansion in the mass ratio $z = M_\mu/M_\tau \approx 6\cdot 10^{-2}$ leading to a power series in z^2.[1] After performing the expansion on the diagram level one is left with at most the calculation of four-loop vacuum diagrams, which have been extensively studied in the literature. To obtain a good numerical accuracy it is sufficient to consider the first three terms in the expansion

$$10^2 A_2^{(4)}(M_\mu/M_\tau) \approx 4.21670 + 0.03257 + 0.00015 = 4.24941(2)(53)\,. \quad (8)$$

The indicated errors correspond to the truncation of the series and to the parametric uncertainty of the mass ratio. We compare the obtained results for the various diagram classes shown in Fig. 1 in Tab. 1. For all classes excellent agreement has been found.

group	$10^2\cdot A_2^{(4)}(M_\mu/M_\tau)$	
	Ref. [12]	Ref. [3]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

Table 1: Results from Ref. [12] in comparison with Ref. [3]. The diagram classes are shown in Fig. 1.

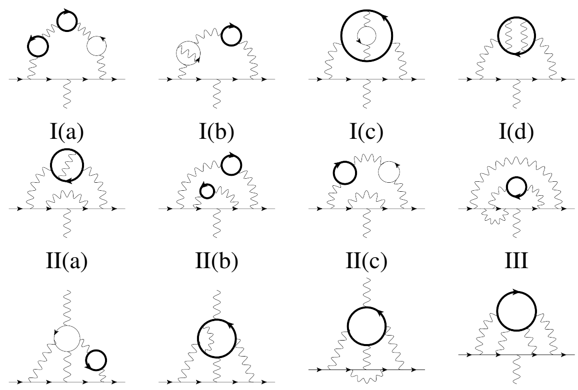

Figure 1: Classes of diagrams considered in Ref. [12]. Thin and thick lines denote light and heavy leptons, respectively.

3. Hadronic vacuum polarization contributions at NNLO

Contributions from the hadronic vacuum polarization are calculated by evaluating

$$a_\mu^{\mathrm{had}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2\int_{m_\pi^2}^{\infty} \mathrm{d}s R(s)K(s) \quad (9)$$

with the R-ratio $R(s) = \sigma(e^+e^- \to \mathrm{hadrons})/\sigma_{\mathrm{pt}}$, $\sigma_{\mathrm{pt}} = 4\pi\alpha^2/(3s)$ and a kernel function $K(s)$, which at leading order is given by

$$K^{(1)}(s) = \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2+(1-x)s/M_\mu^2}\,. \quad (10)$$

[1] Although the expansion is performed in z^2, terms of order z^{2n+1} are generated by performing the mass renormalization in the on-shell scheme.

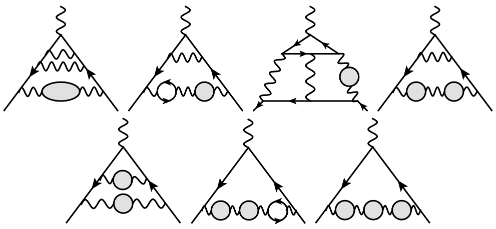

Figure 2: Diagrams contributing to the hadronic vacuum polarization at next-to-next-to-leading order.

The kernel function receives higher-order corrections from perturbation theory. The next-to-leading order corrections to the kernel function have been calculated in [13]. The next-to-next-to-leading order corrections have recently been calculated in [14] by performing an asymptotic expansion in M_μ^2/s. The diagram classes contributing are displayed in Fig. 2. The obtained result for the NNLO contribution

$$a_\mu^{\text{had,NNLO}} = 1.24 \pm 0.01 \times 10^{-10}\,, \tag{11}$$

is larger than expected from lower orders. Taking this contribution into account when comparing theory and experiment reduces the discrepancy by 0.2 standard deviations.

4. Leptonic vacuum polarization contributions at five-loop order

Similar to the hadronic vacuum polarization insertions leptonic ones can be calculated by integrating over the vacuum polarization function $\Pi(q^2)$

$$a_\mu^{\text{lep-vacpol}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\frac{1}{1+\Pi(s_x)}\,,\; s_x = -\frac{x^2}{1-x}M_\mu^2\,. \tag{12}$$

This analysis has been done at four loops in [15]. At five loops the method has first been implemented using only the leading term in the high-energy expansion as approximation for $\Pi(q^2)$[16]. The analysis showed unexpected deviations from the results in [3] and was later improved in [17] where a Padé approximation was used for the vacuum polarization. For the construction of the Padé approximation of the vacuum polarization function at four loops all available information in the low- and high-energy and in the threshold region has been used. In this follow-up analysis the discrepancies were resolved and we compare all three results in Tab. 2. As can be seen there is good agreement between the new analysis and the numerical results for all diagram classes shown in Fig. 3.

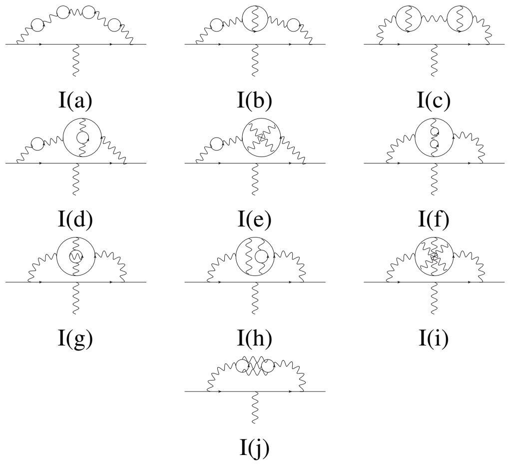

Figure 3: Diagram classes considered in the analysis of Ref. [17].

	Ref. [17]	Ref. [16]	Refs. [18–21]
I(a)	20.142 813	20.183 2	20.142 93(23)
I(b)	27.690 061	27.718 8	27.690 38(30)
I(c)	4.742 149	4.817 59	4.742 12(14)
I(d+e)	6.241 470	6.117 77	6.243 32(101)(70)
I(e)	-1.211 249	-1.331 41	-1.208 41(70)
I(f+g+h)	$4.446\,8^{+6}_{-4}$	4.391 31	4.446 68(9)(23)(59)
I(i)	$0.074\,6^{+8}_{-19}$	0.252 37	0.0 87 1(59)
I(j)	$-1.246\,9^{+4}_{-3}$	-1.214 29	-1.247 26(12)

Table 2: Comparison of the results from Ref. [17], Ref. [16] and Refs. [18–21]. The diagram classes are shown in Fig. 3.

5. Conclusions

We reviewed recent progress in perturbative calculations for the anomalous magnetic moment of the muon. Much progress has been made to further improve the theory prediction. Higher-order corrections to the hadronic vacuum polarisation contribution reduced the difference between experiment and theory by about 0.2 standard deviations.

Acknowledgments

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