Available online at www.sciencedirect.com
ScienceDirect
Nuclear Physics B Proceedings Supplement 00 (2015) 1–27
Nuclear Physics B Proceedings Supplement

Soft-collinear factorization in B decays

M. Beneke
Physik Department T31, James-Franck-Straße 1, Technische Universität München, D–85748 Garching, Germany

Abstract

The combination of collinear factorization with effective field theory originally developed for soft interactions of heavy quarks provides the foundations of the theory of exclusive and semi-inclusive B decays. In this article I summarize some of the later conceptual developments of the so-called QCD factorization approach that make use of soft-collinear effective theory. Then I discuss the status and results of the calculation of the hard-scattering functions at the next order, and review very briefly some of the phenomenology, covering aspects of charmless, electroweak penguin and radiative (semi-leptonic) decays.

Keywords: Flavour physics, B-meson, charmless decays, QCD factorization, CP violation

1. Introduction

In the early 1990s the heavy-quark effective theory [1, 2, 3] and the heavy-quark expansion [4, 5, 6] were developed into powerful tools to describe the dynamics and emerging symmetries [7, 8] of QCD in the limit, when the mass m_b of a quark is much larger than the intrinsic scale Λ of the strong interaction. The basic assumption for the validity of this expansion is that the limit is taken when all other dimensionful parameters, including the momenta of particles, are fixed to be of order Λ or another soft scale. After integrating out the scale m_b, the effective dynamics is governed exclusively by the soft scale, since no other scale is present in the problem.

In the second half of the 1990s the CLEO experiment at Cornell accumulated enough statistics of B mesons to measure for the first time the rare $b \to u$ and loop-induced decays in exclusive two-body final states. The tools above do not apply to this situation. When the final state consists of two light mesons, say two pions, the energy of the pions is roughly $m_b/2$. The presence of a large external momentum invalidates the assumption that all small-virtuality fluctuations are soft. Energetic, light particles can radiate energetic particles and remain close to their mass-shell when the radiation is collinear to their direction of flight. When the invariant mass squared is of order Λ^2 as must be the case for an exclusive final state of light mesons, collinear radiation cannot be described perturbatively.

The idea of collinear factorization is central to the QCD treatment of high-energy scattering with classic applications to deep-inelastic scattering, exclusive form factors at high-momentum transfer, and jet physics. In these cases, a crucial step is often the demonstration that soft effects cancel, such that the non-perturbative, collinear physics can be isolated in parton distributions and light-cone distribution amplitudes. The then novel experimental accessibility of exclusive, energetic final states in B-meson decays required the extension of collinear factorization to situations where the large energy is injected into the process not by a colourless external source such as the electromagnetic current, but the weak decay of a heavy quark surrounded by the soft degrees of freedom that make up the B meson. Soft physics is therefore expected to be more relevant than in traditional applications of collinear factorization.

The problem was first addressed in a systematic way by Buchalla, Neubert, Sachrajda and myself [9, 10, 11], motivated by the search for a rigorous QCD treatment of exclusive heavy-light ($D\pi$) and light-light ($\pi\pi$, "charm-

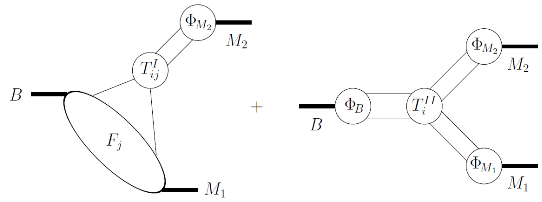

Figure 1: Graphical representation of the factorization formula. Only one of the two form-factor terms in (1) is shown for simplicity. Figure from [10].

less") final states, based on nothing but the scale hierarchy $m_b \gg \Lambda$ and the concepts of perturbative soft and collinear factorization. Assuming that the electroweak scale M_W is already integrated out, the starting point is the weak effective Lagrangian, which contains the flavour-changing interactions in the form of local dimension-six operators Q_i of the four-fermion or chromo- and electromagnetic dipole type. The relevant objects to describe the $B \to M_1 M_2$ decay amplitude are then the matrix elements $\langle M_1 M_2 | Q_i | \bar{B} \rangle$ of these operators. For the further discussion below, I quote the factorization formula, valid in the heavy-quark limit up to corrections of order Λ/m_b, in the form originally given in [9, 10]:

$$\begin{aligned}\langle M_1 M_2 | Q_i | \bar{B} \rangle \;\; = &\\ &\sum_j F_j^{B\to M_1}(m_2^2) \int_0^1 du\, T^{\rm I}_{ij}(u)\, \phi_{M_2}(u) \\ &+ (M_1 \leftrightarrow M_2) \\ &+ \int_0^1 d\xi du dv\, T^{\rm II}_i(\omega, u, v)\, \phi_B(\xi)\, \phi_{M_1}(v)\, \phi_{M_2}(u) \\ &\qquad \text{if } M_1 \text{ and } M_2 \text{ are both light,} \qquad (1)\end{aligned}$$

$$\begin{aligned}\langle M_1 M_2 | Q_i | \bar{B} \rangle \;\; = &\\ &\sum_j F_j^{B\to M_1}(m_2^2) \int_0^1 du\, T^{\rm I}_{ij}(u)\, \phi_{M_2}(u) \\ &\qquad \text{if } M_1 \text{ is heavy and } M_2 \text{ is light.} \qquad (2)\end{aligned}$$

Here $F_j^{B\to M_{1,2}}(m_{2,1}^2)$ denotes a $B \to M_{1,2}$ form factor, and $\phi_X(u)$ is the light-cone distribution amplitude (LCDA) for the quark-antiquark Fock state of meson X. $T^{\rm I}_{ij}(u)$ and $T^{\rm II}_i(\xi, u, v)$ are hard-scattering functions, which are calculable perturbatively in the strong coupling α_s; $m_{1,2}$ denote the light meson masses. Eq. (1) is represented graphically in Figure 1. The presence of a form factor and of LCDAs shows that soft *and* collinear physics is relevant even at the leading order in the heavy-quark expansion.

Eq. (1) applies to decays into two light mesons, for which the spectator quark in the B meson can go to either of the final-state mesons. An example is the decay $B^- \to \pi^0 K^-$. If the spectator quark can go only to one of the final-state mesons, as for example in $\bar{B}_d \to \pi^+ K^-$, we call this meson M_1 and the second form-factor term related to $M_1 \leftrightarrow M_2$ on the right-hand side of (1) is absent. The factorization formula simplifies when the spectator quark goes to a heavy meson [see (2)], such as in $\bar{B}_d \to D^+\pi^-$. In this case the hard interactions with the spectator quark represented by the third term in (1) can be dropped, because they are power-suppressed in the heavy-quark limit. In the opposite situation that the spectator quark goes to a light meson but the other meson is heavy, factorization does not hold.

The simplest case to gain some intuition is $\bar{B}_d \to D^+\pi^-$, when the D meson is also assumed to be parametrically heavy. The spectator quark and other light degrees of freedom in the B meson have to rearrange themselves only slightly to form a D meson together with the charm quark created in the weak $b \to c\bar{u}d$ transition. For the other two light quarks $\bar{u}d$ to form a pion with energy $O(m_b)$, they must be highly energetic and collinear, and in a colour-singlet configuration. Soft interactions decouple from such a configuration and this allows it to leave the decay region without interfering with the D meson formation [12]. The probability of such a special configuration to form a pion is described by the leading-twist pion LCDA $\phi_\pi(u)$. The factorization formula (2) provides the quantitative framework for this discussion, which allows to compute higher-order corrections. For example, if the light quark-antiquark pair is initially formed in a colour-octet state, one can still show that soft gluons decouple at leading order in Λ/m_b, if this pair is to end up as a pion. This implies that the pair must interact with a *hard* gluon, and hence this provides a calculable strong-interaction correction to the basic mechanism discussed above.[1] An important element in demonstrating the suppression of soft interactions, except for those parametrized by the $B \to D$ form factor, is the assumption that the pion LCDA vanishes at least linearly as the longitudinal momentum fraction approaches the endpoints $u = 0, 1$. This assumption can be justified by the fact that it is satisfied by the asymptotic distribution amplitude $\phi_\pi(u) = 6u(1-u)$, which is the appropriate one in the infinite heavy-quark mass limit. If this were not the case, the pion could be formed with larger probability in an asymmetric configuration, in which one constituent has soft momentum of order Λ. Soft gluons would not decouple from this soft

[1] A correction of this type was computed already in [13], but the generality and importance of the result went unnoticed.

constituent, thus spoiling factorization.

The above discussion relies crucially on the spectator quark in the B meson going to the heavy meson in the final state. If, as in the case of a $D^0\pi^0$ final state, the spectator quark must be picked up by the light meson, the amplitude is suppressed by the $B \to \pi$ form factor. But since the D meson's size is of order $1/\Lambda$, the D^0 formation and $B \to \pi$ transition cannot be assumed to not interfere and factorization is violated.

The case of two light final state mesons is the most interesting one. The dominant decay process is indeed the same as for the case of the $D^+\pi^-$ final state, but this implies that the light meson that picks up the spectator quark is formed in a very asymmetric configuration in which the spectator quark carries a tiny fraction Λ/m_b of the total momentum of the light meson. Such a configuration is suppressed by the endpoint behaviour of the LCDA as discussed above, but owing to this suppression there exists a competing process, in which a hard gluon is exchanged with the spectator quark, propelling it to large energy, thus avoiding the endpoint-suppression penalty factor. If the hard gluon connects to the quark-antiquark pair emanating from the weak decay vertex to form the other light meson, this gives rise to the third contribution to the factorization formula (1). This further contribution, called "hard-spectator interaction", involves three LCDAs, including the one of the B meson and resembles the expressions that appear in the theory of exclusive form factors at large momentum transfer [14, 15].

The above discussion reflects the theoretical understanding of QCD factorization in exclusive B decays as of around 2000. The present article summarizes results from a long-term project on soft-collinear factorization in B-meson decays started in 2003. The three following sections are devoted to three different strains of research. The first covers the further conceptual development of QCD factorization which centres around (a) factorization of the $B \to \pi$ form factor and hard-spectator scattering, (b) a calculable example of factorization of endpoint divergences, and (c) factorization at sub-leading power in Λ/m_b for the semi-inclusive semi-leptonic decays $B \to X_u \ell \nu$. The hard-scattering kernels $T^{\rm I}_{ij}(u)$ and $T^{\rm II}_i(\xi, u, v)$ were computed in [9, 10] at $O(\alpha_s)$, which corresponds to the one-loop and tree approximation, respectively. Since then most of the required calculations for the next order have been done and will be discussed in the second part. One of the most remarkable implications of (1), (2) is that the strong phases, which must be present to make direct CP violation observable, are either $O(\alpha_s)$ from loops at leading power, or $O(\Lambda/m_b)$ power-suppressed. The next order therefore also yields the first QCD correction to the direct CP asymmetries, which is usually required to obtain a reliable result. Finally, some aspects of the phenomenology of the factorization approach for charmless and other exclusive decays will be very briefly presented in the third section.

I conclude this introduction with an apology. The circumstances that require the writing of this article imply that it focuses on my own work. It does not do justice to that of many others and does not substitute a review of the subject that remains yet to be written.

2. Conceptual development of QCD factorization

The further conceptual development of QCD factorization in B decays has greatly benefited from the development of soft-collinear effective theory (SCET) [16, 17, 18, 19, 20], the effective Lagrangian formulation of diagrammatic factorization. In turn, the desire to better understand the factorization of exclusive heavy-quark decays has been instrumental in the early development of SCET. For example, the essence of the diagrammatic two-loop analysis [10] that demonstrated that the structure of infrared singularities in the quark representation of the $B \to D\pi$ amplitude is consistent with (2), is elegantly reproduced by the decoupling of soft gluons from the leading-power collinear SCET Lagrangian through a field redefinition by a soft Wilson line, and the subsequent cancellation of the Wilson lines from the colour-singlet current that overlaps with the pion state [21]. This proves factorization to all orders, provided that SCET reproduces all the relevant infrared degrees of freedom, which is the standard assumption. In the following, however, I focus on hard-spectator scattering, where not all the factorization properties are evident in (1), since two perturbative scales m_b and $\sqrt{m_b\Lambda}$ are hidden in the scattering kernel $T^{\rm II}_i(\xi, u, v)$. It turns out that the main issues can already be understood from the factorization of heavy-to-light form factors, so I turn to these first.

2.1. Heavy-to-light form factors

The heavy-to-light form factors parametrize matrix elements of the form $\langle M(p')|\bar{q}\Gamma b|\bar{B}(p)\rangle$, where Γ denotes a Dirac matrix, and we are interested here in the region of large momentum transfer of $O(m_b)$ to the light meson M. The physics contained in this matrix element is surprisingly rich and complex. First attempts compute the large-recoil form factor in terms of LCDAs date back to [22] and are based on the assumption that the light meson is produced in a configuration where the

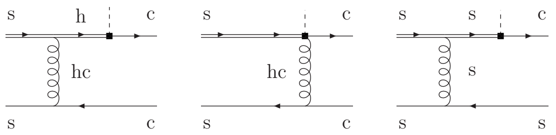

Figure 2: Tree diagrams for the $B \to \pi$ form factor. Left and middle: the pion is produced in a symmetric configuration. In the middle diagram hard fluctuations have already been integrated out. Right: when the pion is produced in an asymmetric configuration, the diagram is purely soft.

Table 1: Scaling of momentum modes relevant to $B \to M$ form factors.

$(n_+p, p_\perp, n_-p)$	terminology	virtuality
$(1, 1, 1)$	hard	1
$(1, \lambda, \lambda^2)$	hard-collinear	λ^2
$(\lambda^2, \lambda^2, \lambda^2)$	soft	λ^4
$(1, \lambda^2, \lambda^4)$	collinear	λ^4

quark and antiquark constituents both carry large momentum, which requires the exchange of an energetic gluon with large virtuality $m_b\Lambda$ (left diagram in Figure 2). However, the calculation exhibits an endpoint divergence, which invalidates this assumption. The opposite starting point was adopted in [23], where it was shown that the three (seven) independent scalar form factors that parametrize the transition of a pseudo-scalar B meson to a pseudo-scalar (vector) light meson can be expressed in terms of only a single (of only two) function(s), if the transition is dominated by soft interactions. This corresponds to the right diagram in Figure 2 and implies that the light meson is always produced in an asymmetric configuration, which though improbable is favoured by the presence of the soft remnant of the B-meson after the heavy-quark decay. Ref. [24] showed that the symmetry relations that leads to the reduction of the number of independent form factors are broken by perturbatively calculable corrections, and that the endpoint divergences that appeared in the hard-scattering approach could be absorbed into the soft form factors. This led to the conjecture that at leading power in the Λ/m_b expansion the scalar form factors factorize as

$$F_i^{BM}(q^2) = C_i(q^2)\,\xi_{BM}(q^2) + \phi_B \otimes T_i(q^2) \otimes \phi_M \quad (3)$$

at large recoil [24]. Here $\otimes$ denotes a convolution in momentum fraction, and $\xi_{BM}(q^2)$ refers to the soft form factor that satisfies the symmetry relations.

With the advent of SCET the heavy-to-light form factors were extensively studied [17, 19, 25, 26, 27, 28] resulting in a "proof" of (3) in [27]. We first note that there are two large scales in the problem. Their existence can already be seen in the left tree diagram in Figure 2. In order to convert the soft spectator quark with momentum Λ into a energetic light quark, the exchanged gluon must have virtuality $m_b\Lambda$. The heavy-quark line that connects the gluon vertex to the external weak vertex, however, is off-shell by the larger amount m_b^2. Factorization should therefore proceed in two steps through the construction of two effective theories, above and below the scale $\sqrt{m_b\Lambda}$ [26]. When the off-shell heavy quark in the left diagram is integrated out, the effective interaction looks as in the middle diagram of Figure 2, which suggests that higher-dimensional operators with additional gluon fields must be relevant [26].

Let the four-momentum of the light meson be $p' = En_-$, where $n_\pm$ define two light-like momenta with $n_+ \cdot n_- = 2$, and $\lambda = \sqrt{\Lambda/m_b}$ defines the small parameter that organizes the power-counting in SCET. The relevant momentum modes are summarized in Table 1. In the B-meson rest frame, the B meson consists of a heavy quark with soft residual momentum and soft constituents, while the light meson is made of collinear partons. The virtuality of soft and collinear modes is Λ^2 and non-perturbative. In contrast, the hard and hard-collinear modes can be integrated out perturbatively one after the other. Each step defines a new effective theory, called SCET_I and SCET_II in [26], and SCET(hc,c,s) and SCET(c,s) in [27]. In the following I summarize the key results in each step, referring to the original papers for many important details. To be definite, M will be assumed to be a pion.

In the first step, the hard modes are integrated out. The SCET Lagrangian is not renormalized to any order [19], but the QCD currents $\bar\psi\Gamma Q$ must be matched to SCET_I. The relevant SCET_I current operators have field content [26]

$$\bar\xi_c h_v, \quad \bar\xi_c A^\mu_{\perp hc} h_v, \quad (4)$$

where here c can be c or hc. The two operators have different λ scaling in SCET_I, but nevertheless both contribute to the leading power in Λ/m_b to the form factor. The reason for this is that SCET_I contains modes of different virtualities, so that the scaling of the matrix elements of (4) is determined only after an analysis of the time-ordered products of the currents with interactions from the SCET_I Lagrangian. Power-suppressed interactions are needed to obtain a term with non-vanishing overlap with the pion and B-meson state, and the analysis shows that both operators types contribute at order $(\Lambda/m_b)^{3/2}$ [27].

Since the (hard-) collinear modes have $n_+\cdot p\sim O(m_b)$, SCET operators are generally non-local and have to be dressed with collinear Wilson lines W_c to make them gauge-invariant. The precise matching equation is

$$(\bar\psi\Gamma_i Q)(0) = \int d\hat s\,\widetilde C_i^{(A0)}(\hat s)\,O^{(A0)}(s;0) + \int d\hat s_1 d\hat s_2\,\widetilde C_{i\mu}^{(B1)}(\hat s_1,\hat s_2)\,O^{(B1)\mu}(s_1,s_2;0) + \cdots, \tag{5}$$

where

$$O^{(A0)}(s;x) \equiv (\bar\xi W_c)(x+sn_+)h_v(x_-) \equiv (\bar\xi W_c)_s h_v, \qquad O_\mu^{(B1)}(s_1,s_2;x) \equiv \frac{1}{m_b}\,(\bar\xi W_c)_{s_1}(W_c^\dagger iD_{\perp c\mu}W_c)_{s_2}h_v \tag{6}$$

correspond to the two operators in (4), and $x_-^\mu = (n_+\cdot x)n_-^\mu/2$, $\hat s_i \equiv s_i m_b/n_- v$. One of the s-integrations can be removed upon taking the matrix elements by using translation invariance. We then define two leading-power $\mathrm{SCET_I}$ pion form factors through

$$\langle\pi(p')|(\bar\xi W_c)h_v|\bar B_v\rangle = 2E\,\xi_{B\pi}(E), \tag{7}$$

$$\langle\pi(p')|\frac{1}{m_b}(\bar\xi W_c)(W_c^\dagger\, i\not{D}_{c\perp}W_c)(rn_+)h_v|\bar B_v\rangle = 2E\int d\tau\, e^{i2E\tau r}\,\Xi_{B\pi}(\tau,E). \tag{8}$$

Here $|\bar B_v\rangle$ denotes the $\bar B$-meson state in the static limit normalized to $2m_B$. At this step, the three pion form factors can be represented as

$$F_i^{B\pi}(E) \;=\; C_i(E)\,\xi_{B\pi}(E) \;+ \int d\tau\, C_i^{(B1)}(E,\tau)\,\Xi_{B\pi}(\tau,E), \qquad i=+,0,T. \tag{9}$$

The point to note here is that the three form factors of the $B\to\pi$ transition can be expressed in terms of a single form factor $\xi_{B\pi}(E)$ and another non-local form factor $\Xi_{B\pi}(\tau,E)$. A number of "symmetry" relations between form factors emerges already at this stage.

The first term of (9) is already in the form of (3) and is often associated with the "soft form factor". While it certainly includes the soft overlap contribution, i.e. the contribution where the pion is formed in the asymmetric configuration (see the right diagram of Figure 2), it is defined as a $\mathrm{SCET_I}$ matrix elements still containing hard-collinear short-distance effects. The matching of the operator $(\bar\xi W_c)_s h_v$ to $\mathrm{SCET_{II}}$ by integrating out the hard-collinear effects has been studied in [27, 28], but has so far been unsuccessful. The problem arises from the absence of a consistent interpretation of endpoint divergences that appear in the convolution integrals of the matching coefficients, which point to missing degrees of freedom. The analysis of [27] shows that the structure must be complicated, since three-particle LCDAs of the B-meson and pion appear at leading power. $\xi_{B\pi}(E)$ is therefore treated as an unknown, non-perturbative function. Fortunately, there is only one instead of the original three (two instead of seven for vector mesons).

Yet (9) would be of little use, if the second term containing an unknown form factor $\Xi_{B\pi}(\tau,E)$ of *two* variables could not be simplified. In the second matching step, one therefore integrates out the hard-collinear scale $\sqrt{m_b\Lambda}$. Note that this step is only applied to the second term and not the first for the reasons mentioned above. A power-counting analysis [27] shows that the operator $O_\mu^{(B1)}\sim\bar\xi_c A^\mu_{\perp hc}h_v$ matches only on four-quark operators of the form $(\bar q_s h_v)(\bar\xi_c\xi_c)$ with no additional fields or derivatives at leading power in the λ expansion. The explicit operator matching relation reads

$$\begin{aligned} &2E\int\frac{dr}{2\pi}\,e^{-i2E\tau r}\,(\bar\xi W_c)(0)(W_c^\dagger\, i\not{D}_{c\perp}W_c)(rn_+)h_v(0) \\ &= \int d\omega dv\, J(\tau;v,\ln(E\omega/\mu^2))\Big[(\bar\xi W_c)(sn_+)\frac{\not{n}_+}{2}\gamma_5(W_c^\dagger\xi)(0)\Big]_{\rm FT}\Big[(\bar q_s Y_s)(tn_-)\frac{\not{n}_-}{2}\gamma_5(Y_s^\dagger h_v)(0)\Big]_{\rm FT} + \ldots \end{aligned} \tag{10}$$

with

$$Q(v) \equiv \Big[(\bar\xi W_c)(sn_+)\frac{\not n_+}{2}\gamma_5(W_c^\dagger\xi)(0)\Big]_{\rm FT} = \frac{n_+p'}{2\pi}\int ds\, e^{-isvn_+p'}\,(\bar\xi W_c)(sn_+)\frac{\not n_+}{2}\gamma_5(W_c^\dagger\xi)(0), \quad (11)$$

$$P(\omega) \equiv \Big[(\bar q_s Y_s)(tn_-)\frac{\not n_-}{2}\gamma_5(Y_s^\dagger h_v)(0)\Big]_{\rm FT} = \frac{1}{2\pi}\int dt\, e^{it\omega}\,(\bar q_s Y_s)(tn_-)\frac{\not n_-}{2}\gamma_5(Y_s^\dagger h_v)(0), \quad (12)$$

and $J(\tau; v, \ln(E\omega/\mu^2))$ the hard-collinear matching coefficient, which can be computed perturbatively. (The middle diagram in Figure 2 is one of the two tree diagrams that contribute to J.) Y_s denotes a soft Wilson line. The Wilson lines W_c, Y_s ensure the gauge-invariance of the light-cone operators.

In the SCET$_{\rm II}$ Lagrangian there are no interactions between collinear and soft fields, since the sum of a collinear and soft momentum has hard-collinear virtuality, which is already integrated out. The SCET$_{\rm II}$ matrix element of a product of collinear and soft fields as it appears on the right-hand side of (10) always falls apart into separate collinear and soft matrix elements. In the present case, these define the LCDA

$$\langle\pi(p')|Q(v)|0\rangle = -if_\pi E\,\phi_\pi(v) \quad (13)$$

for the pion and

$$\langle 0|P(\omega)|\bar B_v\rangle = \frac{i\hat f_B m_B}{2}\,\phi_{B+}(\omega) \quad (14)$$

for the B-meson [24, 29]. Thus the four-fermion operator factorizes into simpler matrix elements upon integrating out the hard-collinear scale and the non-local form factor can be written as [27]

$$\Xi_{B\pi}(\tau, E) = \frac{m_B}{4m_b}\int_0^\infty d\omega\int_0^1 dv\, J(\tau; v, \ln(E\omega/\mu^2))\,\hat f_B\phi_{B+}(\omega) f_\pi\phi_\pi(v). \quad (15)$$

An important point needs to be made here. Due to the absence of soft-collinear interactions in the SCET$_{\rm II}$ Lagrangian *any* operator at any order in the λ expansion formally factorizes into a convolution of a soft and a collinear function, but the result is not always correct, since the convolution integrals often diverge at the endpoints — see the discussion of $\xi_{B\pi}(E)$ above. The final part of the factorization "proof" therefore consists of showing that the convolution integrals in (15) converge. The argument given in [27] shows that J can depend on ω only through terms of the form $1/\omega \times \ln^n(E\omega/\mu^2)$, which together with the endpoint behaviour of the B-meson LCDA guarantees the convergence of the ω-integral. From this the convergence of the v-integral is inferred, since an endpoint divergence in a collinear integral must always have a correspondence in a soft convolution.

The subtleties of soft-collinear factorization are related to the factorization of modes with equal rather than hierarchical virtuality ([27], Figure 2), which are distinguished by the scaling of a longitudinal light-cone momentum component rather than transverse momentum. The factorization of such momentum regions usually leads to divergences that are not regulated in dimensional regularization. Other schemes that work such as analytic regularization whereby one raises certain propagators to non-integer powers $1/[-k^2]^{1+a}$ violate some of the longitudinal boost symmetries of the classical Lagrangian, which constrain the dependence of coefficient functions and operators on light-cone momentum components in the naive factorization formula. The absence of a regulator that preserves all the properties of the classical theory at the quantum level implies that SCET$_{\rm II}$ factorization is generally "anomalous" [30] and valid only in special cases, such as the form factor $\Xi_{B\pi}(\tau, E)$ discussed above. To my knowledge, a general argument when SCET$_{\rm II}$ factorization is *not* anomalous is not available. These difficulties do not affect the first matching step where hard modes are integrated out. At least, I do not know of an example for which the convolution in the hard-collinear momentum fraction (τ above) is divergent or dimensional regularization fails in matching the full theory to SCET$_{\rm I}$.

Returning to the heavy-to-light form factor we plug (15) into (9) and obtain the conjectured factorization formula (3), provided we identify the kernel $T_i(q^2)$ of

$$T_i(E;\nu,\ln\omega) = \frac{m_B}{4m_b}\int d\tau\, C_i^{(B1)}(E,\tau)\,\hat{f}_B f_\pi J(\tau;\nu,\ln(E\omega/\mu^2))\,, \tag{16}$$

or $T_i = C_i^{(B1)} \otimes J$ in short. The hard-scattering kernels of spectator scattering are themselves a convolution of a hard and a hard-collinear function [26]. The one-loop corrections to $C_i^{(B1)}$ [31, 32] and J [33, 34] have both been calculated, and the explicit results confirm the general factorization formula, including the arguments supporting the convergence of the convolution integrals. We therefore conclude that the three independent $B \to \pi$ form factors can be expressed in terms of a single form factor $\xi_{B\pi}$ plus corrections that begin at $O(\alpha_s(\sqrt{m_b\Lambda}))$ and can be expressed in terms of the universal, process-independent LCDAs of the mesons.

2.2. Spectator scattering in hadronic decays

I have discussed the heavy-to-light form factors at length, since after these preparations the extension to charmless B decays is relatively simple. Charmless decays were first considered in SCET in [35, 36]. After the development of the theory of spectator scattering for the form-factor, the derivation of the QCD factorization formula within the SCET framework was completed in [37].

Instead of current operators one needs to match the operators present in the weak effective Hamiltonian. As an example, consider the contribution of $Q = [\bar{u}_b\gamma^\mu(1-\gamma_5)b_a][\bar{d}_a\gamma_\mu(1-\gamma_5)u_b]$ to the decay $\bar{B}_d \to \pi^+\pi^-$ (a, b are colour indices). The extra π^- relative to the $B \to \pi$ transition has large momentum q in the direction opposite to π^+. The SCET Lagrangian is therefore extended by a second collinear sector with fields χ, A_{c2} in addition to ξ, A_{c1}, satisfying $\not{n}_-\xi = 0$ and $\not{n}_+\chi = 0$. Analogous to (5) the operator Q matches to the expression

$$Q = \int d\hat{t}\, \tilde{T}^{\rm I}(\hat{t}) O^{\rm I}(t) + \int d\hat{t} d\hat{s}\, \tilde{H}^{\rm II}(\hat{t},\hat{s}) O^{\rm II}(t,s) \tag{17}$$

at leading power after integrating out the hard modes, where the two operator structures are now given by

$$\begin{aligned} O^{\rm I}(t) &= (\bar{\chi}W_{c2})(tn_-)\frac{\not{n}_-}{2}(1-\gamma_5)(W_{c2}^\dagger\chi)\Big[\tilde{C}_{f_+}^{(A0)}(\bar{\xi}W_{c1})\not{n}_+(1-\gamma_5)h_v \\ &\quad -\frac{1}{m_b}\int d\hat{s}\,\tilde{C}_{f_+}^{(B1)}(\hat{s})\,(\bar{\xi}W_{c1})\not{n}_+[W_{c1}^\dagger i\not{D}_{\perp c1}W_{c1}](sn_+)(1+\gamma_5)h_v\Big], \end{aligned} \tag{18}$$

$$O^{\rm II}(t,s) = \frac{1}{m_b}\Big[(\bar{\chi}W_{c2})(tn_-)\frac{\not{n}_-}{2}(1-\gamma_5)(W_{c2}^\dagger\chi)\Big]\Big[(\bar{\xi}W_{c1})\frac{\not{n}_+}{2}[W_{c1}^\dagger i\not{D}_{\perp c1}W_{c1}](sn_+)(1+\gamma_5)h_v\Big]. \tag{19}$$

It might appear that a final state with two pions should create a major problem, since the physics of charmless decays is obviously more complicated than that of the form factor. However, the two collinear sectors are already decoupled after the first matching step to $\rm SCET_I$, since the sum of a c1 and c2 momentum necessarily has hard virtuality $O(m_b^2)$. Thus there can be no coupling of the two modes in the $\rm SCET_I$ Lagrangian, and hence the $\langle\pi^-\pi^+|\ldots|\bar{B}\rangle$ matrix element of the operators (18), (19) factorizes already at the hard-collinear scale into a $\langle\pi^-|\ldots|0\rangle$ matrix element of the c2 fields and the $\langle\pi^+|\ldots|\bar{B}\rangle$ matrix element of the remainder after decoupling the soft gluons from the c2 fields by the soft Wilson-line field redefinition in the same way as for $\bar{B}_d \to D^+\pi^-$. The former simply gives the pion LCDA $\phi_{M_2}(u)$ in (1). The latter matrix elements are the same as those that appear in the factorization of the $B \to \pi$ form factors. Following [38], the square bracket in (18) has been defined such that its matrix element coincides with the full QCD $B \to \pi$ form factor. This yields the first term in (1). The matrix element of the second square bracket in (19) can be related to $\Xi_{B\pi}$ defined in (8), so (15) can also be used here. Introducing the Fourier transform

$$H^{\rm II}(u,v) = \int d\hat{t}d\hat{s}\, e^{i(u\hat{t}+(1-v)\hat{s})}\,\tilde{H}^{\rm II}(\hat{t},\hat{s}) \tag{20}$$

of the hard-matching coefficient of the operator $O^{\rm II}(t,s)$ associated with the spectator scattering, where u, v correspond to collinear momentum fractions, one finds that the kernel $T^{\rm II}(\omega,u,v)$ in (1) is given by the convolution

$$T^{\rm II}(\omega,u,v) = -\frac{m_B}{8m_b}\int_0^1 dz\, H^{\rm II}(u,z)\, J(1-z;v,\omega). \tag{21}$$

Here J is the same hard-collinear matching coefficient that appeared in the factorization of the form factor. This reproduces the QCD factorization formula (1) for charmless decays in the SCET framework.

The SCET derivation yields several new insights. First, the spectator-scattering kernel is a convolution $T^{\rm II} = H^{\rm II} \otimes J$ of a hard and hard-collinear function, and precise operator definitions can be given, which are important for consistent higher-order calculations (see following section). Second, the decoupling of the second light meson at the hard-collinear scale implies that the perturbative strong interaction phases originate from the imaginary part of hard loops, and never from hard-collinear loops. Third, the upper limit of the ξ-integral in (1) should be replaced by an integral over ω with upper limit ∞ instead of 1. The variable $\xi = \omega/m_B$ corresponds to a light-cone momentum fraction of the spectator quark in the B meson. However, the LCDA $\phi_B(\omega)$ is defined in SCET$_{\rm II}$, which contains a static quark field h_v of formally infinite mass. Relative to this the variable ω can take any value. The one-loop renormalization of $\phi_B(\omega)$ [39] shows that a perturbative tail is generated that ranges to $\omega = \infty$, unless the LCDA is defined in a cut-off scheme. Since the characteristic scale is $\omega \sim \Lambda$, the B-meson LDCA can appear in leading-power factorization formulae only in the form of the inverse moments

$$\sigma_n(\mu) \equiv \lambda_B(\mu)\int_0^\infty \frac{d\omega}{\omega}\,\phi_{B+}(\omega,\mu)\,\ln^n\frac{\mu_0}{\omega}, \tag{22}$$

where the most important moment $\lambda_B(\mu)$ is defined by $\sigma_0(\mu) \equiv 1$. At order $O(\alpha_s^k)$, up to $n = 2k-2$ logarithms can appear, so the number of parameters related to the B-meson LCDA is limited to a few in practical calculations.

2.3. Variants of factorization for hadronic decays

Several variations of factorization as described above have been advocated. The PQCD framework [40, 41] assumes that the B-meson transition form factors $F_i^{BM_1}(0)$ are also dominated by short-distance physics and factorize into light-cone distribution amplitudes. Both terms in (1) can then be combined to

$$\langle M_1M_2|Q_i|\bar{B}\rangle = T^{\rm PQCD} \star \phi_B \star \phi_{M_1} \star \phi_{M_2}. \tag{23}$$

The implicit assumption that the heavy-to-light form factors do not receive any soft contribution seems difficult to justify to me, and certainly contradicts the picture developed above. At the technical level, the endpoint divergences are regularized by keeping the intrinsic transverse momentum of the meson constituents. There is a larger sensitivity to perturbative corrections at low scales, where the strong coupling is large and perturbation theory potentially unreliable. From a phenomenological perspective, PQCD needs fewer non-perturbative input parameters, but there is a larger dependence on unknown light-cone distribution amplitudes. A principal difference between the PQCD and the QCD factorization approach described above is the relative importance of the weak-annihilation mechanism and the generation of strong rescattering phases. In QCD factorization the phases arise at the scale m_b from loop diagrams that have yet to be included in the PQCD approach, and from a model for power-suppressed weak annihilation, whereas in the most widely used implementation of PQCD the strong phases originate only from a weak-annihilation tree diagram. A recent attempt to compute radiative corrections in the PQCD approach resulted in uncancelled infrared divergences [42], which led to the introduction of final-state specific non-perturbative parameters, in violation of factorization. Since a clear power-counting scheme in Λ/m_b has never been established for the PQCD formula (23), this is not necessarily in contradiction with factorization at leading power in Λ/m_b. However, it prevents a systematic improvement of PQCD calculations beyond tree-level.

Ref. [37] proposed a phenomenological implementation of factorization[2] that differs in two important respects from the BBNS approach [9, 10, 11]. First,

[2] Sometimes called the "SCET approach", which I find misleading, since the theoretical basis of diagrammatic QCD factorization and SCET factorization is exactly the same. See the derivation of (1) in the previous subsection.

perturbation theory at the intermediate scale $\sqrt{m_b\Lambda}$ is avoided by not factorizing the spectator-scattering term into a hard and jet function. Eq. (1) then takes the form

$$\langle M_1M_2|Q_i|\bar{B}\rangle = F^{BM_1}\, T_i^{\rm I} \star \phi_{M_2} + \Xi^{BM_1} \star H_i^{\rm II} \star \phi_{M_2}\,, \qquad (24)$$

which is simply the matrix element of the SCET_I representation (17) of the four-quark operator of the weak effective Hamiltonian. Second, penguin diagrams with charm loops, the so-called "charming penguins" [43], are claimed to be non-factorizable, hence non-perturbative at leading order in the Λ/m_b expansion. From the phenomenological perspective the principal difference to the BBNS approach concerns again the generation of strong interaction phases. Since the non-local form factor is unknown, Eq. (24) can be used in practice only at tree level, hence the matrix elements have no imaginary parts. The only exception is the matrix element corresponding to the charm penguin amplitude, which is considered as an unknown complex number (one for each final state, to be fitted to data), and therefore represents the only source of direct CP violation. Weak-annihilation effects are neglected, since they are power-suppressed. A critique of this approach is given in [44]. From the theoretical perspective of factorization, the most important issue raised is the question whether the penguin loops with charm factorize or not. The concern is that for massive quark loops the non-relativistic energy scale m_cv^2 near threshold is of order Λ, and hence there appears to be a sensitivity to non-perturbative scales not present for massless quark loops. The suppression of sensitivity to this region and the associated issue of quark-hadron duality for charm-quark loops has been discussed in [45], which resolves the issue in favour of factorization.

2.4. Endpoint divergences in quarkonium final states

Two-body decays to a charmonium (M_2) and a light meson (M_1) were argued [10] to also factorize according to (1) in the non-relativistic (Coulombic) limit $m_cv^2 \gg \Lambda$, since the size of the charmonium $1/(m_cv)$ is small compared to the wave-length of soft gluons, which therefore decouple. However, when the calculation of the hard-scattering functions was done for P-wave charmonia [46], uncancelled divergences appeared, including an endpoint divergence in spectator scattering, which contradicted the expectation of factorization. In this section I briefly review the result of [47], which explains the origin of the divergences and resurrects factorization at leading order in Λ/m_b. The main conceptual interest in this result derives from the fact that (to my knowledge) it constitutes the only example of a consistent factorization of an endpoint divergence in terms of precisely defined operator matrix elements.

An important property of final states with charmonium when the latter is assumed to be a non-relativistic bound state is that there exist two sets of low-energy scales, m_cv, m_cv^2 related to the non-relativistic expansion, and $\sqrt{m_b\Lambda}$, Λ, related to the collinear expansion and the strong-interaction scale. In the following we assume that $m_c \gg \sqrt{m_b\Lambda} \gg m_cv, m_cv^2 \gg \Lambda$.

Integrating out the scales m_b, m_c leads from QCD to an effective theory that combines SCET_I with the non-relativistic effective theory of QCD for the charm quarks. A QCD operator such as $Q = [\bar{c}_b\gamma^\mu(1-\gamma_5)b_a][\bar{s}_a\gamma_\mu(1-\gamma_5)c_b]$, which contributes to the decay $\bar{B}_d \to \chi_{cJ}\bar{K}^0$, matches to $\text{SCET}_\text{I}\otimes$NRQCD operators which have exactly the same form as (18), (19) except that the part referring to the second light meson has to be replaced by a non-relativistic P-wave colour-singlet rather than a light-cone operator:

$$(\bar\chi W_{c2})(tn_-)\frac{\not{n}_-}{2}(1-\gamma_5)(W_{c2}^\dagger\chi) \quad\to\quad O(^{2S+1}P_J^{(1)}) \equiv \psi_v^\dagger\Gamma_\mu\left(-\frac{i}{2}\right)\overset{\leftrightarrow\mu}{D}_\top\chi_v \qquad (25)$$

with $D_\top^\mu = D^\mu-(v\cdot D)v^\mu$ and Γ_μ a spin matrix. One now finds that the tree-level matching coefficient of the operator $O^{\rm II}$ (19) contains terms of the form $m_c^2/(m_b^2(1-y))$ at tree-level, which are absent for massless quarks. Since $\bar{y}\equiv 1-y$ is the fraction of hard-collinear momentum carried by the gluon field in (19), the divergence at $y=1$ occurs in the region when no momentum is transferred to the spectator quark, which is the endpoint region. Indeed, one then finds that the hard-spectator amplitude contains an integral of the form

$$\mathcal{A}^{\text{hard spectator}}_{B\to H(^{2S+1}P_J)K} \supset \frac{f_K\hat{f}_Bm_B}{m_b\lambda_B}\int_0^1 dy\,\frac{\phi_{\bar{K}}(y)}{\bar{y}^2}, \qquad (26)$$

which diverges at $y=1$.

A crucial point is that the presence of the non-relativistic scales allows additional operators to contribute at leading order in Λ/m_b. New and central to the present discussion are colour-octet operators such as

$$O^{\rm I}_\perp({}^3S_1^{(8)}) = \bar\xi W_c \gamma_{\perp\mu}(1-\gamma_5) T^A h_w\, \psi_v^\dagger \gamma_\top^\mu T^A \chi_v. \quad (27)$$

In the case of charmless decays to two light mesons the matrix elements of colour-octet operators can be non-zero only due to power-suppressed soft-gluon interactions, where soft means momentum of order Λ, thus they can be neglected at leading order in the Λ/m_b expansion. For charmonium, however, the decoupling of gluons with small momentum holds only when the momentum is much smaller than $m_c v^2$; gluons with momentum $m_c v^2$ contribute to the octet operator matrix elements even at leading order in Λ/m_b. These contributions are sub-leading in v, but so are the P-wave operators due to the extra derivative in $O({}^{2S+1}P_J^{(1)})$, hence the gluon-exchange contribution to the S-wave octet operators is relevant at leading order in the velocity expansion to P-wave charmonium production.

Spectator scattering thus consists of hard spectator scattering, where the gluon exchanged between the $c\bar c$ system and the spectator quark in the B meson has energy of order m_b, and $\bar y \sim O(1)$; and soft spectator scattering with gluon energy $O(m_c v^2)$, in which case $\bar y \sim O(v^2) \ll 1$ is naturally in the endpoint region. Soft spectator scattering leaves the charm quarks close to their mass-shell and therefore does not reduce to an effective interaction of the form (19). It is part of the colour-octet matrix element to be computed within the effective theory and is sensitive to the charmonium bound-state dynamics. The explicit calculation in the Coulombic limit $m_c v^2 \gg \Lambda$ gives an expression that contains

$$\mathcal{A}^{\rm soft\ spectator}_{B\to H({}^{2S+1}P_J)K} \supset \frac{f_K \hat f_B m_B}{m_b \lambda_B} \int_0^1 dy\, \phi_{\bar K}(y) \left(\sqrt{-\left(\bar y + \frac{2\sqrt{z} E_H}{m_b(1-z)}\right)} + \frac{\gamma_B}{m_b\sqrt{1-z}}\right)^{-4}, \quad (28)$$

where $E_H < 0$ is the binding energy of the charmonium state, $z = 4m_c^2/m_b^2$ and $\gamma_B = (m_c\alpha_s C_F)/2$ is the inverse Bohr radius. We now compare the integral over the kaon distribution amplitude to (26). While the integrand there was applicable to y not near 1 and exhibited a logarithmic endpoint divergence as $y \to 1$, the integrand of (28) is appropriate only to $1-y \sim v^2$, i.e. *in* the endpoint region. There is no divergence here as $y \to 1$. However, for $\bar y \gg v^2$ the integrand has the same logarithmic behaviour $\int dy\, \phi_{\bar K}(y)/\bar y^2$ as does the hard-spectator contribution for $\bar y \ll 1$. Eq. (26) is based on approximations that are invalid when $\bar y$ is small, which causes the endpoint divergence. We can regulate the divergent integral by cutting off the y integral above $1-\mu$. This corresponds to a hard factorization scale in the energy of the gluon that connects to the spectator quark. The spectator-scattering contribution to the colour-octet matrix element originates precisely from the energy region that is then cut out in (26), and is not valid when $\bar y \sim O(1)$, thus the correct interpretation of the y-integral in (28) is $\int_0^1 dy \to \int_{1-\mu}^1 dy$. To combine with (26) we must evaluate the regularized version of (28) up to terms of order v^2/μ. We then find

$$\mathcal{A}^{\rm soft\ spectator}_{B\to H({}^{2S+1}P_J)K} \supset \frac{f_K \hat f_B m_B}{m_b \lambda_B}\, \phi'_{\bar K}(1) \Big(-\ln\mu - \ln(1-z) + 1 + i\pi + \Delta F\Big), \quad (29)$$

where ΔF is a real number given in [47] depending on m_b, m_c, E_H and γ_B. The endpoint contribution is proportional to $\phi'_{\bar K}(1)$, the derivative of the kaon LCDA at the endpoint, because the endpoint region is of size v^2 rather than Λ/m_b, hence it is justified to describe the quarks in the kaon by collinear quark fields.

There are several remarkable features of this result. First, although (28) comes from a tree-diagram contribution to the octet matrix element, it carries a sizable absorptive part related to rescattering, which originates from the on-shell charm propagator in the integral over the charmonium bound-state wave function. Second, the factorization scale dependence on μ cancels exactly with a corresponding logarithm in the (regulated) hard spectator contribution (26). The cancellation occurs as usual between an infrared (endpoint) divergence from the coefficient function and an ultraviolet (large energy) divergence of the octet matrix element. It further occurs between objects that have precise operator definitions in the relevant effective field theory.

Unfortunately, these results do not readily generalize to final states of two light mesons. In the charmonium case above, there are two endpoint regions, $\bar{y} \sim v^2$ and $\bar{y} \sim \Lambda/m_b$, and we showed the cancellation of the endpoint divergence in the former, while still working at leading order in Λ/m_b. One difficulty with endpoint factorization at $\bar{y} \sim \Lambda/m_b$ is that a light meson in the asymmetric $q\bar{q}$ configuration cannot be made out of generic collinear and soft modes in SCET_{II}, since the sum of collinear and soft momenta has virtuality $m_b\Lambda$, not Λ^2. It seems that SCET_{II} is lacking some of the relevant degrees of freedom to write down operators that can describe the endpoint region and overlap with the light-meson wave function. The understanding of this issue seems central to me to further theoretical progress on exclusive B decays, which has to address the factorization properties of power corrections in Λ/m_b. The present use of factorization in exclusive B decays combines a rigorous theory at leading power in Λ/m_b with a rudimentary parametrization of some power corrections, which introduces a considerable but presently unavoidable arbitrariness in some quantities, especially direct CP asymmetries, where power-suppressed phases compete with loop-induced ones.

2.5. Power corrections to semi-inclusive B decay

While a rigorous theory of power-suppressed effects is not available for exclusive B decays, there is one important class of processes for which soft-collinear factorization has been worked out completely including the $O(\Lambda/m_b)$ terms. These are the inclusive semi-leptonic decays $\bar{B} \to X_u\ell\bar{\nu}$, where X_u denotes a hadronic final state without charm, and inclusive radiative and electroweak penguin decays $\bar{B} \to X_s\gamma$, $\bar{B} \to X_s\ell^+\ell^-$ in the kinematic region where the hadronic final-state is collimated into a single jet, which carries large energy of $O(m_b)$, but has small (though not too small) invariant mass $m_b\Lambda \ll m_b^2$. Leading-power factorization into hard, jet and soft (here called shape) functions for these processes was discussed in [17, 48] in the SCET and diagrammatic framework.

The reason why a rigorous treatment of power corrections is possible here is the absence of collinear physics at the scale Λ. The two-step matching at the hard and hard-collinear scale then corresponds to the sequence QCD $\to$ SCET_I $\to$ HQET of effective theories. The jet functions containing the collinear physics at the scale $\sqrt{m_b\Lambda}$ can be computed perturbatively in the strong coupling, while the non-perturbative soft physics is contained in well-defined HQET matrix elements. There is no other mode with virtuality Λ^2 than the soft mode, when the virtuality of the inclusive hadronic final state is $O(m_b\Lambda)$. The effective-theory framework provides a transparent book-keeping of the relevant interactions at every step in the calculation, including power-suppressed effects, and allows us to identify all the relevant HQET operators. The factorization of semi-inclusive decays at $O(\Lambda/m_b)$ has been discussed in [49, 50, 51] at various levels of detail. The discussion below outlines the main points following the presentation in [51], which contains further details and the explicit tree-level computation.

The strong interaction effects in semi-leptonic B decays are contained in the hadronic tensor $W^{\mu\nu}$, related by the optical theorem to the imaginary part of the forward scattering amplitude, which we define as

$$W^{\mu\nu} = \frac{1}{\pi}\,\text{Im}\,\langle\bar{B}(v)|T^{\mu\nu}|\bar{B}(v)\rangle. \tag{30}$$

The correlator $T^{\mu\nu}$ is the time-ordered product of two flavour-changing weak currents

$$T^{\mu\nu} = i\int d^4x e^{-iq\cdot x}\text{T}\{J^{\dagger\mu}(x), J^\nu(0)\}, \tag{31}$$

where q is the momentum carried by the outgoing lepton pair in the $\bar{B} \to X_u\ell\bar{\nu}$ decay and $J^\mu = \bar{q}\gamma^\mu(1-\gamma_5)b$.

The matching of the current J_μ from QCD to SCET_I is the same as was discussed for exclusive decays. Due to rotational invariance in the transverse plane, a frame can be chosen such that the hard-collinear transverse momenta of order $\lambda = \sqrt{\Lambda/m_b}$ appear only in the internal integrations over the momenta in the jet, and since the integral over an odd number of transverse momenta either vanishes or must be proportional to one of the external soft transverse momenta of order λ^2, the resultant expansion is in powers of $\lambda^2 \sim 1/m_b$ rather than power of λ. We therefore need to extend (5) to second order in the SCET expansion parameter λ. The basis of SCET_I heavy-light current operators at $O(\lambda^2)$ contains 15 different operator structures, which are listed in [51]. A few representative examples are

$$\begin{aligned} J_3^{(2)} &= (\bar{\xi}W_c)_s\, i\overleftarrow{\partial}^\mu_\perp\Gamma_j(x_\perp D_s h_v) \\ J_8^{(2)} &= (\bar{\xi}W_c)_{s_1}\, i\overleftarrow{\partial}^\mu_\perp[W_c^\dagger iD^\nu_{\perp c}W_c]_{s_2}\Gamma_j h_v \end{aligned} \tag{32}$$

and

$$\begin{aligned} J_{12}^{(2)} &= (\bar\xi W_c)_{s_1}\Big\{[W_c^\dagger iD_{\perp c}^\mu W_c]_{s_2},[W_c^\dagger iD_{\perp c}^\nu W_c]_{s_3}\Big\}\Gamma_j h_v \\ J_{14}^{(2)} &= \Big[(\bar\xi W_c)_{s_1}\Gamma_j h_v\Big]\Big[(\bar\xi W_c)_{s_2}\frac{\not n_+}{2}\,\Gamma_{j'}(W_c^\dagger\xi)_{s_3}\Big], \end{aligned} \tag{33}$$

where $\Gamma_j = \{1,\gamma_5,\gamma_{\alpha_\perp}\}$ denotes a basis of the four independent Dirac matrices between $\bar\xi$ and h_v. The last two operators contain collinear field products at three independent light-cone positions $x + s_i n_+$, $i = 1, 2, 3$. The momentum-space coefficient functions of these currents therefore depend on two independent variables that describe the distribution of the total light-cone momentum among the three collinear fields. These operators describe an effective heavy-to-light transition vertex with two additional transverse hard-collinear gluons (J_{12}), or an additional hard-collinear quark-antiquark pair (J_{14}) plus any number of $n_+\cdot A_c$ gluon fields from the hard-collinear Wilson line W_c. The soft fields including h_v are multipole-expanded and depend only on x_-^μ. At this point, we can write the representation of the QCD current in SCET$_\mathrm{I}$ to any order in the λ expansion in the form

$$(\bar\psi\Gamma_i Q)(x) = e^{-im_b v\cdot x}\sum_{j,k}\widetilde C_{ij}^{(k)}(\hat s_1,\ldots,\hat s_n)\otimes J_j^{(k)}(\hat s_1,\ldots,\hat s_n;x), \tag{34}$$

where the ⊗ stands for a convolution over a set of dimensionless variables $\hat s_i \equiv s_i m_b$. The superscript k refers to the scaling of the current operator with λ relative to the leading-power currents. The subscript j enumerates the effective currents at a given order in λ. The two-point correlator (31) of QCD currents is accordingly written as

$$T^{\mu\nu} = \tilde H_{jj'}(\hat s_1,\ldots,\hat s_n)\otimes T_{jj'}^{\mathrm{eff},\mu\nu}(\hat s_1,\ldots,\hat s_n). \tag{35}$$

The hard function $\tilde H_{jj'}$ is a product of SCET Wilson coefficients $\tilde C_{ij}^{(k)}\tilde C_{i'j'}^{(k')}$ and $T_{jj'}^{\mathrm{eff},\mu\nu}$ is a correlator of two of the above SCET$_\mathrm{I}$ currents.

The effective correlators are computed with the SCET$_\mathrm{I}$ Lagrangian including the power-suppressed interactions up to $O(\lambda^2)$, which have been derived in [19, 20]. The power-suppressed SCET$_\mathrm{I}$ Lagrangian terms are dealt with in the interaction picture. Writing

$$\mathcal{L}_{\mathrm{SCET_I}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \ldots, \tag{36}$$

the possible time-ordered products that build up the hadronic tensor at $O(\lambda^2)$ are

$$\begin{aligned} &a) \quad J^{(0)}J_k^{(2)} + \text{sym.} \\ &b) \quad J_k^{(1)}J_l^{(1)} \\ &c) \quad J^{(0)}J_k^{(1)}\mathcal{L}^{(1)} + \text{sym.} \\ &d) \quad J^{(0)}J^{(0)}\mathcal{L}^{(2)} \\ &e) \quad J^{(0)}J^{(0)}\mathcal{L}^{(1)}\mathcal{L}^{(1)}. \end{aligned} \tag{37}$$

The $O(\lambda)$ suppressed interactions in the effective Lagrangian involving hard-collinear (ξ) and soft (q_s) quarks read

$$\mathcal{L}_\xi^{(1)} = \bar\xi x_\perp^\mu n_-^\nu W_c\, gF_{\mu\nu}^s W_c^\dagger\frac{\not n_+}{2}\xi, \tag{38}$$

$$\mathcal{L}_{\xi q}^{(1)} = \bar q_s W_c^\dagger i\not D_{\perp c}\xi - \bar\xi i\overleftarrow{\not D}_{\perp c}W_c q_s. \tag{39}$$

At second order there are several terms. Examples are

$$\mathcal{L}_{2\xi}^{(2)} = \frac{1}{2}\bar\xi x_\perp^\mu x_{\perp\rho}n_-^\nu W_c[D_{\perp s}^\rho, gF_{\mu\nu}^s]W_c^\dagger\frac{\not n_+}{2}\xi, \tag{40}$$

and

$$\mathcal{L}_{\mathrm{HQET,mag}}^{(2)} = \frac{C_{\mathrm{mag}}}{4m_b}\,\bar h_v\sigma_{\mu\nu}gF_s^{\mu\nu}h_v, \tag{41}$$

where $C_{\mathrm{mag}} \neq 1$ represents the renormalization of the chromomagnetic interaction by hard quantum fluctuations. The complete list of relevant terms in given in [51].

It is worth noting that the interaction $\mathcal{L}_{\xi q}^{(1)}$ in (39) describes the process $q_c \to g_c + q_s$, that is, the radiation of a soft quark from an energetic quark, which is thereby seen to be power-suppressed relative to the emission of soft gluons, $q_c \to q_c + g_s$. $\mathcal{L}_{\xi q}^{(1)}$, despite being power-suppressed, is a crucial term for hard-spectator scattering at *leading* order in the $1/m_b$ expansion discussed in previous sections, since it turns the soft spectator quark in the B meson into the energetic quark that forms the constituent of an energetic light meson. $\mathcal{L}_\xi^{(1)}$, on the other hand, describes the correction to the soft-gluon emission process $q_c \to q_c + g_s$, and hence the leading correction to the usual soft ("eikonal") approximation. Due to the vanishing of odd powers in the λ expansion, even the second-order corrections in λ are needed to obtain the Λ/m_b term to the semi-leptonic decay rate, which includes a time-ordered product with two insertions of $\mathcal{L}^{(1)}$ in (37).

By construction the time-ordered products (37) are evaluated with the leading-order Lagrangian $\mathcal{L}^{(0)}$. We recall that when the hard-collinear fields are redefined according to [18]

$$\xi = Y\xi^{(0)}, \quad A_c = YA_c^{(0)}Y^\dagger, \quad W_c = YW_c^{(0)}Y^\dagger, \tag{42}$$

where Y is the soft Wilson line of $n_- \cdot A_s$, the soft and collinear fields are decoupled in $\mathcal{L}^{(0)}$. The $\bar{B}$ meson by definition contains no collinear degrees of freedom, meaning that the $\bar{B}$-meson state is represented as the tensor product $|\bar{B}\rangle \otimes |0\rangle$, where the first (second) factor refers to the soft (hard-collinear) Hilbert space. It follows that the matrix element of any SCET_I current correlator $T^{\text{eff},\mu\nu}$, including in general time-ordered products with sub-leading interactions from the Lagrangian, can be written in the factored form

$$\langle \bar{B}|T^{\text{eff}}(\hat{s}_1,\ldots,\hat{s}_n)|\bar{B}\rangle = i\int d^4x d^4y \ldots e^{i(m_b v-q)x}\,\langle \bar{B}|\bar{h}_v[\text{soft fields}]h_v|\bar{B}\rangle(x_-,y_-,...)$$
$$\times\,\langle 0|[\text{hard}-\text{collinear fields}]|0\rangle(\hat{s}_1,\ldots,\hat{s}_n;x,y,...). \tag{43}$$

The additional integrals over $d^4y\ldots$ are related to insertions of the power-suppressed Lagrangian terms. The soft matrix element depends only on $x_+ \equiv n_+x, y_+ \equiv n_+y,\ldots$, so the integrations over transverse positions and the n_- components can be lumped into the definition of the collinear factor. The soft and collinear matrix elements are then linked by multiple convolutions over the light-cone variables $x_+, y_+,\ldots$. Having carried out these steps, and using momentum-space representations of the hard coefficients, jet-functions, and shape-functions, a generic term in the factorization formula is a sum of convolutions

$$T = \sum H(u_1,\ldots u_i)\otimes \mathcal{J}(u_1,\ldots,u_i;\omega_1,\ldots,\omega_n)$$
$$\otimes\, S(\omega_1,\ldots,\omega_n). \tag{44}$$

to any order in the λ expansion. The convolution variables $u_i = n_+\cdot p_i/m_b$ are the fractions of longitudinal momentum $n_+\cdot p_i$ carried by the hard-collinear fields in SCET_I. The convolution variable ω_i is conjugate to $n_+\cdot x_i$, and corresponds to $n_-\cdot k_i$, where k_i is the (outgoing) momentum of a soft field.

The jet functions are defined as the vacuum matrix elements of hard-collinear fields as they appear in (43). Since the hard-collinear virtuality $m_b\Lambda \gg \Lambda^2$, the jet functions can be computed in perturbation theory. On the other hand, the soft B-meson matrix elements in (43) define non-perturbative functions. At any given order in the λ expansion, the soft functions are obtained by stripping the hard-collinear fields (since they define the jet functions) from the time-ordered product terms. Up to $O(\lambda^2)$, one is left with the B-meson matrix elements

$$\langle \bar{B}|(\bar{h}_vY)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle, \quad \langle \bar{B}|(\bar{h}_vY)(x_-)_{a\alpha}(Y^\dagger iD_s^\mu Y)(z_-)_{cd}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle,$$
$$\langle \bar{B}|(\bar{h}_vY)(x_-)_{a\alpha}(Y^\dagger iD_{s\perp}^\mu Y)(z_{1-})_{cd}(Y^\dagger iD_{s\perp}^\nu Y)(z_{2-})_{ef}(Y^\dagger h_v)(0)_{b\beta}|\bar{B}\rangle,$$
$$\langle \bar{B}|(\bar{h}_vY)(x_-)_{a\alpha}(Y^\dagger h_v)(0)_{b\beta}\,(\bar{q}_sY)(z_{1-})_{c\gamma}(Y^\dagger q_s)(z_{2-})_{d\delta}|\bar{B}\rangle, \tag{45}$$

where colour (Latin) and spinor (Greek) indices have been made explicit. These matrix elements are then decomposed in colour and spin into scalar functions, which depend on one, two or three ω variables. It follows that to order $1/m_b$, but to arbitrary order in α_s in the coefficient functions, the semi-inclusive semi-leptonic differential decay rate depends on a large number of multi-local shape-functions. An investigation of the structure of the convolution shows that at this order the full-QCD hadronic tensor has the factorized structure

$$T = H\cdot \mathcal{J}(\omega)\otimes S(\omega)$$

$$+ \sum H(u_1, u_2) \otimes \mathcal{J}(u_1, u_2; \omega) \otimes S(\omega) + \sum H(u) \otimes \mathcal{J}(u; \omega_1, \omega_2) \otimes S(\omega_1, \omega_2)$$
$$+ \sum H \cdot \mathcal{J}(\omega_1, \omega_2, \omega_3) \otimes S(\omega_1, \omega_2, \omega_3) + \ldots, \qquad (46)$$

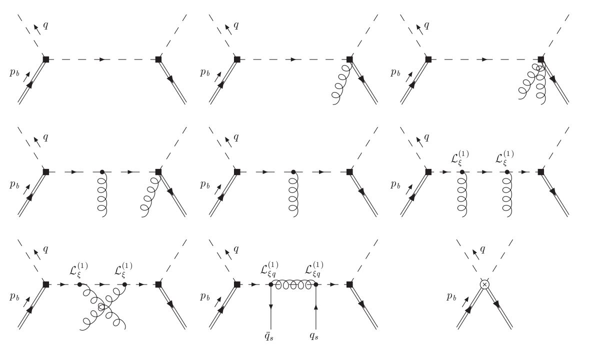

Figure 3: Tree diagrams contributing to the current correlators $T^{\mathrm{eff},\mu\nu}$. Not shown are diagrams that vanish when $n_+A_c = 0$, $n_-A_s = 0$, or are symmetric to those shown. External lines (except for the dashed current insertion) are soft. The double lines refer to the heavy-quark field h_v. Figure from [51].

where the ellipses denote $1/m_b^2$ terms not considered here, and for each term the dependence on the convolution variables of the most complicated structure is shown. The first line represents the leading order in the Λ/m_b expansion. It depends only on a single function of a single variable. It is evident that the Λ/m_b correction is much more complicated. It is unlikely that multi-dimensional soft functions can ever be reliably determined from data or be computed by non-perturbative methods, so some amount of modelling of these functions will be required. The renormalization-group equations of the effective theory allow large logarithms $\ln m_b/\Lambda$ to be summed to all orders. The anomalous dimensions are themselves multi-dimensional integration kernels. There is a considerable number of them, given the number of different terms, which may all mix under renormalization. The problem of summing logarithms to all orders at $O(1/m_b)$ has therefore so far not been addressed.

Fortunately, the structure of the result is much simpler than (46) in the tree approximation as was shown in [50, 51]. Figure 3 displays the relevant tree diagrams. The reason for this simplification is that tree diagrams can only have cuts with a single internal hard-collinear particle, and hence there are no convolutions in hard-collinear variables u_i. The convolutions of multi-dimensional soft functions with the tree-level jet functions can be lumped into a set of unknown functions of a *single* variable. An interesting aspect of this analysis is that there is a contribution from a four-quark HQET operator of the form $\bar{h}_v h_v \bar{q}_s q_s$ to the differential semi-leptonic decay rate at $O(1/m_b)$ (see the middle diagram in the last row of Figure 3), whereas in the total decay rate such contributions are at least of order $1/m_b^3$. For a given flavour of q_s, these operators contribute differently to the decay of the charged and the neutral B meson, primarily in the region of small $n_- \cdot P$, where P is the total momentum of the hadronic final state X_u. For a numerical estimate, which requires some modelling, I refer to [51].

Note that the operator $T^{\mu\nu}$ in (31) appears in similar form in many other processes. The SCET formalism described here can be adapted to address power corrections in the process $e^+e^- \to$ two jets, in semi-inclusive deep-inelastic scattering (DIS) $e^- + p \to e^- + X$, when the hadronic state X is collimated to form a single jet, or Drell-Yan production near the threshold. For DIS, the analogy is especially clear, since one only needs to replace the B-meson by a proton, and the weak current by the electromagnetic current. SCET provides the expressions for the sub-leading interactions and hard vertices that are needed to extend the standard factorization theorems based on the eikonal approximation and the factoring of jet functions to sub-leading power.

3. Radiative corrections

Significant efforts have been dedicated over the past ten years to push factorization calculations to $O(\alpha_s^2)$ in the perturbative expansion. This implies two-loop calculations of the matching coefficient of the $\mathrm{SCET_I}$ current $\bar{\xi} h_v$ and the form-factor kernel T^{I} for hadronic two-body decays, and one-loop calculations of the hard- and hard-collinear matching coefficients in the spectator-scattering terms. In this section I review these efforts and summarize a few results.

3.1. Form-factor relations

The matching coefficients relevant to the factorization of the heavy-to-light form factors can be read off from (9), (15), (16). At $O(\alpha_s^2)$ one needs the one-loop corrections to $C_i^{(B1)}$, the coefficient functions of the $\bar{\xi}_c A_{\perp hc} h_v$ operators, and to J, defined by the non-local operator

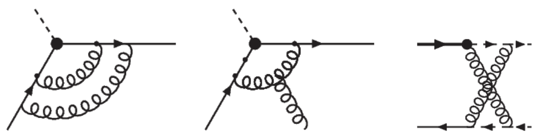

Figure 4: Representative diagrams for the two-loop contribution to C_i, and the one-loop contribution to $C_i^{(B1)}$ and J (from left to right).

matching equation (10). The two pieces were obtained in [31, 32], and [33, 34], respectively. In addition, one needs the two-loop coefficient functions C_i of the $\xi_c h_v$ operators, which have been calculated in [52, 53, 54, 55] for the vector and axial-vector QCD current and in [56] for the remaining (pseudo) scalar and tensor currents. Since all quantities involved have operator definitions, these are fairly standard loop calculations with special vertices derived from the form of the operators and the SCET Lagrangians. Figure 4 shows a representative diagram for each of the three calculations.

With these results the symmetry-breaking corrections to the heavy-to-light form factor relations [23] that were first computed at $O(\alpha_s)$ in [24], can be extended to $O(\alpha_s^2)$. This has interesting applications to exclusive and semi-inclusive semi-leptonic and radiative B decays [56], of which a few will be mentioned here.

Two examples of form-factor ratios, which simplify in the heavy-quark limit, are

$$\begin{aligned} \mathcal{R}_1(E) &\equiv \frac{m_B}{m_B+m_P}\frac{f_T(E)}{f_+(E)}, \\ \mathcal{R}_2(E) &\equiv \frac{m_B+m_V}{m_B}\frac{T_1(E)}{V(E)}. \end{aligned} \tag{47}$$

We can write the first in the form

$$\mathcal{R}_1(E) = R_T(E) + \int_0^1 d\tau\, C_{T+}^{(B1)}(\tau,E)\,\frac{\Xi_P(\tau,E)}{f_+(E)}, \tag{48}$$

where $R_T = C_{f_T}/C_{f_+}$ is the ratio of the two-loop matching coefficients for the tensor and vector currents for the $B \to P$ (pseudo-scalar) transition, $C_{T+}^{(B1)}(\tau,E) = C_{f_T}^{(B1)}(\tau,E) - C_{f_+}^{(B1)}(\tau,E)\,R_T(E)$, and $\Xi_P(\tau,E)$ is given in (15). A similar result holds for the second ratio $\mathcal{R}_2(E)$ that applies to $B \to V$ (vector) form factors. We recall that factorization as discussed here is valid when the energy of the outgoing light meson satisfies $E \gg \Lambda$. In this limit, the ratios above approach the value 1 plus perturbative corrections comprised in $R(E)$, which are independent of any non-perturbative parameters, and the spectator-scattering correction. Since this term begins at $O(\alpha_s)$, the sensitivity to non-perturbative form factors and light-cone distribution amplitudes is reduced to $O(\alpha_s)$ in ratios.

To be specific, consider the point of maximal recoil, i.e. $E = m_B/2$ or $q^2 = 0$, at which we find

$$\begin{aligned} R_T(E_{\rm max}) &= 1 + \frac{\alpha_s^{(4)}(\mu)}{4\pi}\left[\frac{8}{3} - \frac{4}{3}L_\nu\right] + \left(\frac{\alpha_s^{(4)}(\mu)}{4\pi}\right)^2 \left[-\frac{100}{9}L_\mu L_\nu + \frac{200}{9}L_\mu + 6L_\nu^2 - \frac{922}{27}L_\nu \right. \\ &\quad \left. -\frac{16}{3}\zeta(3) + \frac{10}{3}\pi^4 - \frac{952}{27}\pi^2 + \frac{8047}{162} + \frac{128}{27}\pi^2\ln 2\right], \end{aligned} \tag{49}$$

with $L_\mu = \ln(\mu^2/m_b^2)$, $L_\nu = \ln(\nu^2/m_b^2)$. The scale ν is used to distinguish the scale dependence of the QCD tensor current from the residual scale dependence of the truncated perturbative expansion. Numerically, the form factor ratio $\mathcal{R}_1(E_{\rm max})$ is (setting $\nu = \mu = m_b$) [56]

$$\begin{aligned} \mathcal{R}_1(E_{\rm max}) &= 1 + \Big[0.046\,(\text{NLO}) + 0.015\,(\text{NNLO})\Big]\,(R_T) \\ &\quad -0.160\Big\{1 + 0.524\,(\text{NLO spec.}) - 0.002\,(\delta^{\parallel}_{\rm log})\Big\} \;=\; 0.817. \end{aligned} \tag{50}$$

In this expression we separated the symmetry-conserving term (first number, equal to 1), the correction to R_T (remaining terms in the first line) and the spectator-scattering contribution (second line, including a small correction $\delta^{\parallel}_{\rm log}$ from renormalization-group summation [34]), and within these the $O(\alpha_s^2)$ contributions "NNLO" and "NLOspec.".

We observe that these $O(\alpha_s^2)$ corrections are 30% (first line of (50)) and 50% (second line) of the $O(\alpha_s)$ ones, and result in an overall reduction of $\mathcal{R}_1(E_{\rm max})$

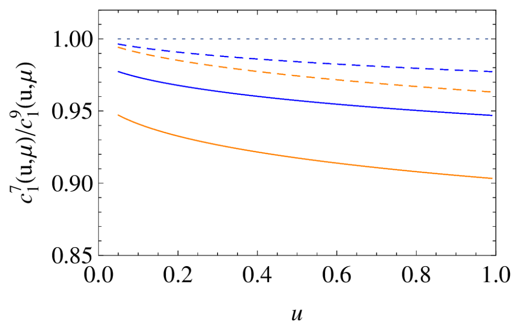

Figure 5: The matching coefficient $c_1^7(u,\mu)/c_1^9(u,\mu)$ as a function of u (related to the di-lepton invariant mass $q^2 = (1-u)m_b^2$) in the one-loop (dashed) and two-loop (solid) approximation. The blue/dark grey curves refer to $\mu = m_b = 4.8\,$GeV, and the orange/light grey ones to $\mu = 1.5\,$GeV. Figure from [56].

by 20% relative to the symmetry limit. The R_T and spectator-scattering corrections have opposite sign, but the latter are larger and determine the sign of the deviation from the symmetry limit. The same observations hold for $\mathcal{R}_2(E_{\max})$, but in this case the sign of the two terms is opposite and one finds $\mathcal{R}_2(E_{\max}) = 1.067$. The normalization of the spectator-scattering term (the number -0.160 in (50)) involves the parameter combination

$$r_{\rm sp} = \frac{9 f_\pi \hat{f}_B}{m_b f_+^{B\pi}(0)\lambda_B}, \qquad (51)$$

which captures most of the dependence on non-perturbative parameters of spectator scattering and the form-factor ratio as a whole, and supplies the by far dominant source of quantifiable theoretical uncertainty. In addition there are $O(\Lambda/m_b)$ corrections to the form-factor factorization formula (9).

There are also QCD sum-rule calculations of the QCD form factors, which have different theoretical systematics, and implicitly include some $O(\Lambda/m_b)$ power corrections. These calculations give [57, 58] $\mathcal{R}_1(E_{\max}) = 0.955$ and $\mathcal{R}_2(E_{\max}) = 0.947$. For the tensor-to-vector ratio $\mathcal{R}_2$, one notices that the sign of the symmetry-breaking correction is opposite for these two methods. The origin of this discrepancy is not understood. Before drawing conclusions on the sum-rule method or the size of power corrections, however, a dedicated analysis of form-factor ratios (rather than the form factors themselves) with correlated theoretical uncertainties should be performed within the QCD sum-rule method.

3.2. Semi-inclusive $B \to X_s \ell^+\ell^-$ decay

The two-loop correction to the QCD current matching to SCET$_{\rm I}$ has an interesting application to semi-inclusive $B \to X_s \ell^+\ell^-$ decay. As discussed in Section 2.5, the semi-inclusive rate factorizes in the form

$$d\Gamma^{[0]} = h^{[0]} \times J \otimes S\,, \qquad (52)$$

at leading order in the $1/m_b$ expansion with hard functions $h^{[0]}$ that can be expressed in terms of the coefficients C_i of $\xi_c \Gamma_i h_v$.

Here I focus on the differential (in q^2, the $\ell^+\ell^-$ invariant mass) forward-backward (FB) asymmetry in the angle between the positively charged lepton and the $\bar{B}$ meson in the centre-of-mass frame of the di-lepton pair in the presence of an upper limit on the invariant mass m_X of the hadronic final state [59, 60]. It is well-known that due to the interplay of electroweak penguin operators mediating $b \to s\ell^+\ell^-$ and the electromagnetic dipole operator mediating $b \to s\gamma(\to \ell^+\ell^-)$, the FB asymmetry exhibits a zero. In the presence of an invariant-mass cut $m_X^{\rm cut}$ the location q_0^2 of the zero is given approximately by [56]

$$\frac{q_0^2}{2m_b(m_B - \langle p_X^+\rangle)} = -\frac{{\rm Re}\,[C_7^{\rm incl}(q_0^2)]}{{\rm Re}\,[C_9^{\rm incl}(q_0^2)]}\,\frac{c_1^7(u_0)}{c_1^9(u_0)} \qquad (53)$$

with $u_0 \equiv 1 - q_0^2/(m_b(m_B - \langle p_X^+\rangle))$ and $\langle p_X^+\rangle \approx p_X^{+\rm cut}/2$. This result depends on $m_X^{\rm cut}$ through the corresponding cut on the light-cone momentum component $p_X^{+\rm cut}$.

The first factor on the right-hand side of (53) is primarily related to the Wilson coefficients of the electroweak penguin and electromagnetic dipole operators in the weak effective Lagrangian. The second factor $c_1^7(u)/c_1^9(u)$ provides a modification from the matching to SCET$_{\rm I}$ and can be expressed in terms of the coefficients C_i known to the two-loop order. The dependence of this factor on the di-lepton invariant mass is shown at LO, NLO and NNLO in Figure 5 for two different renormalization scales. The impact of the NLO correction is to shift q_0^2 by -2.2%. The size of the NNLO correction is significant, in fact larger than the NLO correction, and shifts the zero by another -3%. Including an estimate of power-suppressed effects, Ref. [56] finds

$$q_0^2 = \left[(3.34\ \ldots\ 3.40)^{+0.22}_{-0.25}\right]{\rm GeV}^2 \qquad (54)$$

for $m_X^{\rm cut} = (2.0\ldots 1.8)\,$GeV. This value of the asymmetry zero in semi-inclusive $b \to s\ell^+\ell^-$ decay is significantly smaller than for the exclusive case (see below), where spectator scattering is responsible for a positive shift as is the fact that in this case $\langle p_X^+\rangle = 0$ in (53). On the other hand the semi-inclusive zero is in the same region as in the inclusive case [61].

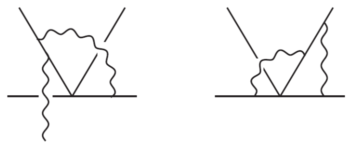

Figure 6: Representative diagrams for the one-loop contribution to $H^{\rm II}$ (left), and the two-loop contribution to $T^{\rm I}$ (right).

3.3. Tree-dominated charmless hadronic decays

The matching coefficients relevant to the factorization of the charmless, hadronic, two-body decays can be read off from (1), (17), (21). At $\mathcal{O}(\alpha_s^2)$ one needs the one-loop correction to $H^{\rm II}$, the coefficient function of the $(\bar\chi\chi)(\bar\xi_c A_{\perp hc} h_v)$ spectator-scattering operator $O^{\rm II}$, and the two-loop correction to $T^{\rm I}$ related to the $(\bar\chi\chi)(\xi_c h_v)$ operator $O^{\rm I}$. The hard-collinear function J is the same that enters the form factor. The computation of $H^{\rm II}$ at one-loop was performed in [38] for the so-called tree-operators $Q_{1,2}$ in the weak effective Lagrangian. Two independent calculations can be found in [62, 63]. The two-loop matching of $Q_{1,2}$ to the SCET$_{\rm I}$ four-quark operator, which defines $T^{\rm I}$, was first done for the imaginary part of the amplitude [64] and then in two independent calculations [65, 66] for the full amplitude, finding complete agreement. These results together extend the $\mathcal{O}(\alpha_s)$ result of [9, 11] to the next order for pure tree-operator decays such as $B^-\to\pi^-\pi^0$. Figure 6 shows a representative diagram for each of the two calculations.

The calculations are more involved than those for the form factors, since the presence of the second light meson adds another variable to the functional dependence, such that $H^{\rm II}$ depends on two momentum fractions, see (20). Furthermore, the four-fermion operators consist of two strings of fermion lines between which gluons can be exchanged. When factorization is done in dimensional regularization, one has to deal with the renormalization of operators that vanish in four dimensions ("evanescent operators"), whose matrix elements must be renormalized to zero. Related to this is the issue of consistent Fierz transformations in d dimensions. The transition $b\to u(\bar u d)$, which is effected by the local operators $Q_{1,2}$ and where the bracket indicates the fermion fields which are contracted, can contribute to the $\pi^+\pi^-$ final state in the form $(\bar q_s u)(\bar u d)$ (called "right insertion") as well as to the $\pi^0\pi^0$ final state in the form $(\bar q_s d)(\bar u u)$ (called "wrong insertion"), where $q_s = d$ is the spectator quark in the $\bar B_d$ meson. In the second case a Fierz rearrangement of the original fermion strings is required to match to the SCET$_{\rm I}$ operators (18), (19). The simplest way to deal with this difficulty is to add the difference of Fierz-equivalent operators to the list of evanescent operators to be renormalized to zero [38, 66]. In the following I briefly elaborate on the issue of evanescent operators and the subtractions required to arrive at the renormalized coefficient functions.

In case of $H^{\rm II}(u,v)$, one has to calculate the one-loop five-point amplitudes $b\to q_{c2}\bar q_{c2} q_{c1} g_{c1}$ associated with the matrix elements $\langle q(q_1)\bar q(q_2) q(p_1') g(p_2')|Q_{1,2}|b(p)\rangle$, where the collinear light particles carry momentum $q_1 = u m_b n_+/2$, $q_2 = \bar u m_b n_+/2$ in collinear-2 direction, and $p_1' = v m_b n_-/2$, $p_2' = \bar v m_b n_-/2$ in the other. When one strips the SCET$_{\rm I}$ operator $O^{\rm II}(t,s)$ off all its fields, and represents it only by its Dirac structure,

$$O_1 \equiv \frac{\not n_-}{2}(1-\gamma_5)\otimes\frac{\not n_+}{2}(1-\gamma_5)\gamma_\perp^\mu, \qquad (55)$$

one finds that one-loop right-insertion amplitude also contains the operators

$$O_n = \frac{\not n_-}{2}\gamma_\perp^\mu\gamma_\perp^{\alpha_1}\dots\gamma_\perp^{\alpha_{n-1}}(1-\gamma_5)\otimes\frac{\not n_+}{2}(1-\gamma_5)\gamma_{\perp\alpha_1}\dots\gamma_{\perp\alpha_{n-1}} \qquad (56)$$

with $n = 2,3,4$. All these operators are evanescent, i.e. vanish in four dimensions. They disappear from the final result, since we shall renormalize them such that their infrared-finite matrix elements vanish, but they must be kept in intermediate steps, hence the matching equation (17) has to be extended to include all four operators on the right-hand side.

Evanescent operators appear already at tree level. In this approximation the right-insertion matrix element of Q_2 is given by

$$\langle Q_2\rangle_{\rm nf} = \frac{1}{N_c}\left(\frac{2}{\bar u}\langle O_1\rangle - \frac{1}{u\bar u}\langle O_2\rangle\right). \qquad (57)$$

The subscript "nf" (for "non-factorizable") means that the "factorizable" diagrams with no lines connecting the collinear-2 and collinear-1 sectors are omitted, since they belong to the $T^{\rm I}O^{\rm I}$ term in (17) by virtue of the definition (18). While one can simply set $\langle O_2\rangle = 0$ in the above equation, since no $1/\epsilon$ poles are present at tree level, the appearance of an evanescent operator in the d-dimensional tree level matrix element implies that one must compute the mixing of O_2 into O_1 in the 1-loop calculation. The renormalized coefficient function of the physical (non-evanescent) operator $O^{\rm II}$ can then be written as

$$\begin{aligned} H^{\rm II(1)} &= A_1^{(1)} - A_1^{(0)} * Z_{11}^{(1)} - A_2^{(0)} * Z_{21}^{(1)} \\ &\quad + 2\,T^{\rm I(1)} C_{f_+}^{(B1)(0)}, \end{aligned} \tag{58}$$

where $A_n^{(k)}$ denotes the coefficient of O_n of the non-factorizable contribution to the ultraviolet-renormalized QCD k-loop $b \to q_{c2}\bar{q}_{c2}q_{c1}g_{c1}$ amplitude. The first term is the one-loop amplitude, which is infrared divergent. The next term is the SCET_I $\overline{\text{MS}}$ counterterm $Z_{11}^{(1)}$ for the physical operator times its tree coefficient $A_1^{(0)}$, which subtracts this divergence. The third term arises from the mixing of the evanescent operator O_2 into O_1 at one-loop. The finite renormalization constant $Z_{21}^{(1)} = -M_{21}^{(1)\rm off}$ can be computed from the mixing matrix element with an infrared regularization different from the dimensional one (for instance, off-shell). Finally, the last term arises from the fact that we defined (18) to reproduce the full QCD form factor. The stars in (58) remind us that the amplitudes depend on momentum fractions, so the renormalization "constants" are actually convolution kernels. For the wrong insertion coefficient function, the expression is slightly more complicated and contains an additional term

$$\tilde{A}_1^{(0)} * \left(\tilde{M}_{11}^{(1)\rm off} - \tilde{M}_{00}^{(1)\rm off}\right). \tag{59}$$

It is finite and independent of the infrared regulator and ensures that the difference of Fierz-equivalent SCET_I operators is correctly renormalized to zero. Furthermore, due to the Fierz rearrangement there could be a contribution

$$\begin{aligned} &\tilde{A}_{1,\rm f}^{(1)} - A_{1,\rm f}^{(1)} + \tilde{A}_{2,\rm f}^{(0)} * \tilde{M}_{21}^{(1)\rm off} \\ &+ \tilde{A}_{1,\rm f}^{(0)} * \left(\tilde{M}_{11}^{(1)\rm off} - \tilde{M}_{00}^{(1)\rm off}\right), \end{aligned} \tag{60}$$

from factorizable diagrams, which is not contained in the definition of the QCD form factor. However, this term vanishes here.

The calculation of $T_i^{\rm I}(u)$ amounts to evaluating the two-loop on-shell Q_i matrix element of the transition $b(p) \to q_1(p')q_2(uq)\bar{q}_3(\bar{u}q)$ with kinematics $p = p' + q$, $p^2 = m_b^2$, $p'^2 = q^2 = 0$. The ultraviolet and infrared finite short-distance coefficient is obtained from

$$\begin{aligned} T_i^{(0)} &= A_{i1}^{(0)}, \\ T_i^{(1)} &= A_{i1}^{(1)\rm nf} + Z_{ij}^{(1)} A_{j1}^{(0)}, \\ T_i^{(2)} &= A_{i1}^{(2)\rm nf} + Z_{ij}^{(1)} A_{j1}^{(1)} + Z_{ij}^{(2)} A_{j1}^{(0)} + Z_\alpha^{(1)} A_{i1}^{(1)\rm nf} \\ &\quad + (-i)\,\delta m^{(1)} A_{i1}'^{(1)\rm nf} \\ &\quad - T_i^{(1)}\left[C_{FF}^{(1)} + Y_{11}^{(1)} - Z_{ext}^{(1)}\right] \\ &\quad - \sum_{b>1} H_{ib}^{(1)} Y_{b1}^{(1)}, \end{aligned} \tag{61}$$

where now $A^{(k)}$ denotes the *bare* k-loop QCD amplitude, Z_{ij}, $\delta m^{(1)}$ the QCD renormalization matrices including those for the operators Q_i and the bottom mass counterterm, Y_{ab} the SCET_I renormalization factors, and $H_{ib}^{(1)}$ the one-loop coefficients of evanescent SCET_I four-quark operators. Again convolutions are implied, for instance, $H_{ib}^{(1)} Y_{b1}^{(1)}$ must be interpreted as the convolution product $\int_0^1 du'\, H_{ib}^{(1)}(u')\, Y_{b1}^{(1)}(u', u)$. The corresponding equation for the two-loop wrong insertion matrix element is significantly more complicated, as the additional terms from Fierz rearrangement do not vanish any more, and the difference of two-loop SCET_I counterterms must be computed [66].

The computation of the bare two-loop QCD matrix elements $A_{i1}^{(2)\rm nf}$ is technically the most challenging part of (61) and has been done by reducing tensor integrals to scalar integrals by Passarino-Veltman reduction [67], the reduction of scalar integrals to master integrals using the Laporta algorithm [68] based on integration-by-parts (IBP) identities [69, 70], and finally by evaluating the master integrals directly, or through Mellin-Barnes representations in the more complicated cases. It is convenient to compute the convolutions

$$\int_0^1 du\; 6u(1-u) C_n^{(3/2)}(2u-1)\, T_i^{(k)}(u) \tag{62}$$

of the kernels with the first few terms in the Gegenbauer expansion

$$\phi_M(u) = 6u\bar{u}\left[1 + \sum_{n=1}^{\infty} a_n^M\, C_n^{(3/2)}(2u-1)\right] \tag{63}$$

of the light-meson LCDA to express the final result in terms of the first two Gegenbauer moments $a_{1,2}^M$. In [66] fully analytic expressions after integration over the Gegenbauer expansion, including the exact dependence on the charm-quark mass, have been obtained.

Putting these results together, we can proceed to the investigation of the so-called topological tree amplitudes with $O(\alpha_s^2)$ accuracy. For definiteness, only the three $B \to \pi\pi$ decay modes will be considered. The decay amplitudes are given by

$$\sqrt{2}\,\mathcal{A}_{B^-\to\pi^-\pi^0} \;=\; A_{\pi\pi}\lambda_u\Big[\alpha_1 + \alpha_2\Big],$$

$$\mathcal{A}_{\bar{B}^0\to\pi^+\pi^-} = A_{\pi\pi}\left\{\lambda_u\Big[\alpha_1+\hat{\alpha}_4^u\Big]+\lambda_c\,\hat{\alpha}_4^c\right\},$$
$$-\mathcal{A}_{\bar{B}^0\to\pi^0\pi^0} = A_{\pi\pi}\left\{\lambda_u\Big[\alpha_2-\hat{\alpha}_4^u\Big]-\lambda_c\,\hat{\alpha}_4^c\right\}, \quad (64)$$

where $A_{\pi\pi} \equiv i\,G_F m_B^2 f_\pi f_+^{B\pi}(0)/\sqrt{2}$ and $\lambda_p = V_{pb}V_{pd}^*$. The dominant amplitudes governing the set of $\pi\pi$ amplitudes are the so-called colour-allowed (colour-suppressed) tree amplitude $\alpha_1(\pi\pi)$ ($\alpha_2(\pi\pi)$) and the non-singlet QCD penguin amplitudes $\alpha_4^p(\pi\pi)$. The equation does not show some smaller amplitudes that are taken into account in the numerical evaluation of the branching fractions below. (Full expressions for the amplitudes are given in [71].) The normalization of the topological tree amplitudes is $\alpha_1 = 1$ and $\alpha_2 = 1/3$ in the approximation by tree-level W-boson exchange, when renormalization-group evolution of the Q_i from the electroweak scale to m_b is neglected. The NNLO QCD factorization results for the $\pi\pi$ final states read [66]

$$\begin{aligned}\alpha_1(\pi\pi) &= 1.009+[0.023+0.010\,i]_{\rm NLO}+[0.026+0.028\,i]_{\rm NNLO}\\ &\quad-\left[\frac{r_{\rm sp}}{0.445}\right]\Big\{[0.014]_{\rm LOsp}+[0.034+0.027i\,]_{\rm NLOsp}+[0.008]_{\rm tw3}\Big\} = 1.000^{+0.029}_{-0.069}+(0.011^{+0.023}_{-0.050})i\,, \quad (65)\end{aligned}$$

$$\begin{aligned}\alpha_2(\pi\pi) &= 0.220-[0.179+0.077\,i]_{\rm NLO}-[0.031+0.050\,i]_{\rm NNLO}\\ &\quad+\left[\frac{r_{\rm sp}}{0.445}\right]\Big\{[0.114]_{\rm LOsp}+[0.049+0.051i\,]_{\rm NLOsp}+[0.067]_{\rm tw3}\Big\} = 0.240^{+0.217}_{-0.125}+(-0.077^{+0.115}_{-0.078})i\,. \quad (66)\end{aligned}$$

The first line of each of the two equations corresponds to the contribution from the operators $(\bar{\chi}\chi)(\bar{\xi}_c h_v)$ with coefficient function $T^{\rm I}$, while the second line accounts for spectator scattering at $O(\alpha_s)$ ("LOsp") and $O(\alpha_s^2)$ ("NLOsp") and a certain power correction ("tw3"). The numerical result sums all contributions and provides an estimate of the theoretical uncertainties from the parameter variations as detailed in [66].

The amplitudes $\alpha_{1,2}$ are scale- and scheme-independent physical quantities. Figure 7 shows the residual renormalization scale dependence at LO, NLO and NNLO of the vertex correction $T^{\rm I}$ to the colour-suppressed amplitude α_2. The NLO contributions to this amplitude are large, because they are proportional to the tree operator with large Wilson coefficient in the weak effective Lagrangian. The scale dependence stabilizes for the real part at NNLO, while the absorptive part receives another sizable correction, approximately 25% of the leading-order real part, and hence its scale-dependence is barely reduced. By comparing the theoretical uncertainty of the full expressions in (65), (66) above to the one from the vertex correction alone, see Figure 7, we see that it arises primarily from spectator scattering. The main contributors to the uncertainty are the parameter combination (51), which appears as an overall normalization factor of the spectator-scattering term, the second Gegenbauer moment $a_2^\pi(2\,\text{GeV})$, and the "tw3" estimate the power correction.

We observe that the vertex and spectator-scattering corrections come with opposite sign, both at NLO and NNLO, and separately for the real and imaginary part. This seems to be a general pattern that was also observed for the form factor ratios discussed in the previous subsection. Here it is somewhat unfortunate, since an enhancement rather than a cancellation in the colour-suppressed tree amplitude would have been welcome in view of the trend indicated by experimental data, as will be seen below.

The colour-suppressed tree amplitude $\alpha_2(\pi\pi)$ in particular exhibits an interesting structure. It starts out with a positive real value 0.220 at LO. After adding the perturbative corrections to the four-quark vertex (the first line of (66)), it is found to be almost purely imaginary, $0.01 - 0.13i$. Then the spectator-scattering mechanism regenerates a real part of roughly the original size and cancels part of the strong phase. The net result of this is that the colour-suppressed tree amplitude can become sizable in QCD factorization when $r_{\rm sp}$ is large, but since this enhances the cancellation of the imaginary part, one cannot have both, a large magnitude and a large strong phase. In essence, the dynamics of the colour-suppressed tree amplitude is completely governed by quantum effects, and the theoretical uncertainty is correspondingly large. In comparison, the colour-allowed tree amplitude $\alpha_1(\pi\pi)$ is rather stable against radiative corrections, and never deviates by a large amount from

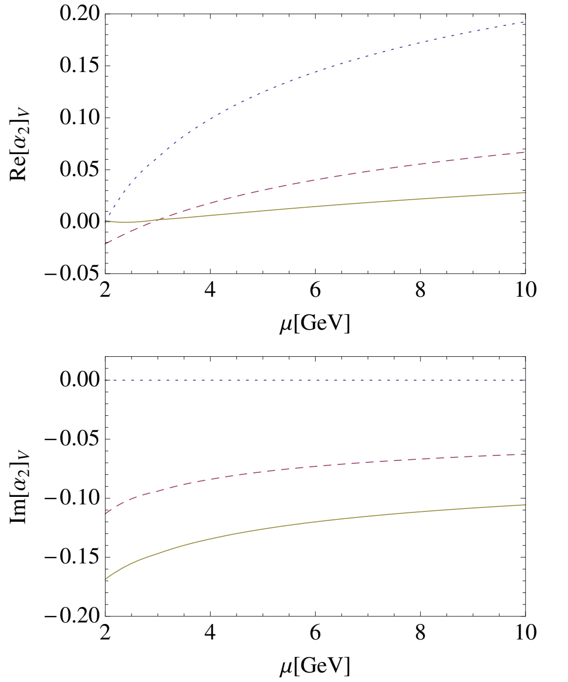

Figure 7: Dependence of the topological tree amplitude $\alpha_2(\pi\pi)$ on the hard scale μ (vertex correction $T^{\rm I}$ only). The dotted, dashed and solid lines refer to the theoretical predictions at LO, NLO and NNLO, respectively. Figure from [66].

its LO estimate.

Strictly speaking, only the charged B-meson tree-operator decays to pions and longitudinally polarized rho mesons can presently be predicted with complete NNLO accuracy, since the penguin amplitude α_4^p that appears in the other two modes of (64) is not yet completely known at $O(\alpha_s^2)$ (see next subsection). By using the best available approximations for α_4^p the full set of $\pi\pi$, $\pi\rho_L$ and $\rho_L\rho_L$ final states has been analyzed in [66, 72]. Table 2 shows an excerpt of the predicted CP-averaged branching fractions compared to present measurements compiled from BABAR and BELLE (including CDF and LHCb for $\pi^+\pi^-$) data as given in [73]. Within uncertainties the agreement between theory and measurement is very good for the final states with a substantial contribution from α_1. The colour-suppressed amplitude α_2 appears to be predicted too small, although the comparison recently improved due to a (preliminary) revision of the previous BELLE measurement. The uncertainties in Table 2 are strongly correlated among the three decay modes. Further information can be brought to light by considering ratios of branching fractions sensitive to certain parameters and amplitudes. I refer to [66, 72] for the details of this analysis and results for final states containing rho mesons.

	Theory	Experiment
$B^- \to \pi^-\pi^0$	$5.82^{+0.07\,+1.42}_{-0.06\,-1.35}$ $(\star)$	$5.48^{+0.35}_{-0.34}$
$\bar{B}^0_d \to \pi^+\pi^-$	$5.70^{+0.70\,+1.16}_{-0.55\,-0.97}$ $(\star)$	5.10 ± 0.19
$\bar{B}^0_d \to \pi^0\pi^0$	$0.63^{+0.12\,+0.64}_{-0.10\,-0.42}$	1.17 ± 0.13

Table 2: CP-averaged branching fractions (BrAv) in units of 10^{-6} of $B \to \pi\pi$ decays. The first error on a quantity comes from the CKM parameters, while the second one stems from all other parameters added in quadrature. Theory corresponds to Theory II in [66] and adopts the values $f_+^{B\pi}(0) = 0.23 \pm 0.03$, $\lambda_B(1\,\text{GeV}) = (0.20^{+0.05}_{-0.00})\,\text{GeV}$ for the pion form factor and B-meson LCDA parameter, respectively. In order to focus on the hadronic uncertainty, the ranges given refer to the uncertainty of $\text{BrAv}(\bar{B} \to f)/|V_{ub}|^2$. Ranges marked by an asterisk $(\star)$ handle the dependence on the heavy-to-light form factors in a similar way and refer to $\text{BrAv}(\bar{B} \to f)/(|V_{ub}|^2 f_+^{B\pi}(0)^2)$.

3.4. Penguin-dominated charmless hadronic decays

Direct CP asymmetries require the interference of amplitudes with different CKM factors, λ_u and λ_c in (64), as well as a phase difference of the corresponding hadronic matrix elements. This necessitates the computation of the matrix elements of the so-called penguin operators Q_{3-10} and dipole operators $Q_{7\gamma}$, Q_{8g} in the weak effective Lagrangian.

At $O(\alpha_s^2)$ the one-loop coefficients $H_i^{\rm II}$ and two-loop coefficients $T_i^{\rm I}$ for Q_{3-10} (one loop less for $Q_{7\gamma}$ and Q_{8g}) are needed. They involve, in addition to diagram topologies identical to those relevant to $Q_{1,2}$ such as shown in Figure 6, the "penguin contractions" of same-flavour $q\bar{q}$ fields from the operators Q_i. On the one hand, the penguin contractions are topologically simpler than the vertex diagrams, on the other hand the presence of an internal massive charm or bottom quark loop introduces another dimensionless ratio of scales, which complicates the analytic calculation of the master integrals. The calculation of the hard spectator-scattering kernels for the non-singlet QCD and the colour-allowed and colour-suppressed electroweak penguin amplitudes has been performed in [74], but the two-loop calculation of $T_i^{\rm I}$ has not yet been completed. Technical results on the relevant master integrals have already appeared [75]. Thus, at present, the $O(\alpha_s)$ results of [9, 11, 71] for the penguin amplitudes and direct CP violation cannot be extended to the next order.

I therefore refrain from a detailed discussion of the numerical impact of the spectator-scattering contribution (see [74]) except for one remark. In principle, the one-loop correction to α_4^c can be expected to be rather large, since there is a contribution from the tree operator Q_1 with coefficient $C_1 \sim 1$ ten times larger than the coefficients C_{3-6} of the QCD penguin operators. However, there is an almost complete, possibly accidental cancellation between the contributions from Q_1 with colour factors $C_F = 4/3$ and $C_A = 3$, respectively.

4. Phenomenology

Since the focus of this article is on the development of factorization and radiative corrections, and since the phenomenology of large sets of final states is not yet possible at NNLO in the absence of the calculation of at least the QCD penguin amplitude with this accuracy, this section will be brief, drawing in part on a summary prepared for [76], highlighting what in my judgement are interesting conclusions from and applications of $O(\alpha_s)$ (NLO) calculations.

4.1. Charmless, hadronic two-body decays

Charmless, hadronic two-body decays provide by far the most observables due to the sheer number of possible final states. Two-body here includes final states with unstable particles like kaons, rho mesons etc. as long as they can be clearly identified as sharp resonances. NLO QCD factorization results are available for a variety of complete sets of final states: two pseudo-scalar mesons (PP) and pseudo-scalar plus one vector meson (PV) [71], two vector mesons (VV) [77], final states with a scalar meson [78], an axial-vector meson [79, 80], and a tensor meson [81].

A key issue for phenomenology is the treatment of power corrections, since factorization as embodied by (1) is not expected to hold at sub-leading order in $1/m_b$. Some power corrections related to scalar currents, which appear after Fierz rearrangements, are enhanced by large ("chirally enhanced") factors such as $m_\pi^2/((m_u + m_d)\Lambda)$. Some corrections of this type, in particular those related to scalar penguin amplitudes nevertheless appear to be calculable and turn out to be important numerically. On the other hand, attempts to compute sub-leading power corrections to hard spectator-scattering in perturbation theory usually result in infrared divergences, which signal the breakdown of factorization. These effects are then estimated and included into the error budget, see the contributions marked "tw3" in (65), (66). All weak annihilation contributions belong to this class of effects and often constitute the dominant source of theoretical error, in particular for the direct CP asymmetries. Factorization as above applies to pseudo-scalar flavour-non-singlet final states and to the longitudinal polarization amplitudes for vector mesons. Final states with η, η' require additional considerations, but can be included [82]. The transverse helicity amplitudes for vector mesons are formally power-suppressed but can be sizable [83], and they do not factorize in a simple form [77]. The description of polarization is therefore more model-dependent than branching fractions and CP asymmetries. Besides these conceptual uncertainties, the lack of precise knowledge of quantities such as $|V_{ub}|$, heavy-to-light form factors, and the B-meson LCDA parameter λ_B cause a significant theoretical uncertainty.

Much of the phenomenology of the most widely considered PP, PV and VV final states is determined by the non-singlet *QCD penguin amplitude* α_4^p. This amplitude is certainly underestimated at NLO in α_s and leading order in the heavy-quark expansion. The power-suppressed but chirally-enhanced scalar penguin amplitude, and probably a (difficult to disentangle) weak annihilation contribution is required to explain the penguin-dominated PP final states. While the scalar penguin amplitude is calculable, some uncertainty remains. An important observation is the smaller size of the PV, VP and VV penguin amplitudes as compared to PP final states, which can be inferred from the measured branching fractions of hadronic $b \to s$ transitions. This is a clear indication of the relevance of factorization, which predicts this pattern as a consequence of the quantum numbers of the operators Q_i. If the penguin amplitude were entirely non-perturbative, no pattern of this form would be expected. A similar statement applies to the $\eta^{(\prime)}K^{(*)}$ final states, where factorization explains naturally the strikingly large differences in branching fractions, including the large $\eta' K$ branching fraction, through the interference of penguin amplitudes, although sizable theoretical uncertainties remain [82]. The flavour-singlet penguin amplitude seems to play a sub-ordinate role in these decays.

The situation is much less clear for the *strong phases and direct CP asymmetries*. A generic qualitative prediction is that the strong phases are small, since they arise through either loop effects ($\alpha_s(m_b)$) or power corrections (Λ/m_b). Enhancements may arise, when the leading-order term is suppressed, for instance by small Wilson coefficients. This pattern is indeed observed. Quantitative predictions have met only partial success. The observed direct CP asymmetry in the $\pi^+\pi^-$ and the asymmetry difference in the $\pi^0 K^+$, $\pi^- K^+$ final states are prominently larger than predicted. A comparison of all CP asymmetry results shows a presently ununderstood pattern of quantitative agreements and disagreements. Since $\alpha_s(m_b)/\pi$ and Λ/m_b are roughly of the same order, it is quite possible that power corrections are $O(1)$ effects relative to the perturbative calculation, preventing a reliable quantitative estimate. However, the direct CP asymmetry calculations are still LO calculations, contrary to the branching fractions, so the final verdict must await the completion of the NLO asymmetry calculation, which requires the two-loop computation of α_4^p. Contrary to direct CP asymmetries, the S parameter that appears in time-dependent CP asymmetries is predicted more reliably, since it does not require the computation of a strong phase. This is exploited successfully in NLO computations of the difference between $\sin 2\beta$ from $b \to s$ penguin-dominated and from $b \to c\bar{c}s$ tree decays [84, 85], and the direct determination of the CKM angle γ (α) from time-dependent CP asymmetry measurements for the $\pi^+\pi^-$, $\pi^\pm\rho^\mp$ and $\rho_L\rho_L$ final states ([71] and Figure 1 of [86]).

Polarization in $B \to VV$ decays was expected to be predominantly longitudinal, since the transverse helicity amplitudes are Λ/m_b suppressed due to the V-A structure of the weak interaction and helicity conservation in short-distance QCD. While this is parametrically true (with one exception, see below), closer inspection shows that the parametric suppression is hardly realized in practice for the penguin amplitudes [83]. This leads to the qualitative prediction (or rather, in this case, postdiction) that the longitudinal polarization fraction should be close to 1 in tree-dominated decays, but can be much less, even less than 0.5, in penguin-dominated decays, as is indeed observed. However, quantitative predictions of polarization fractions for penguin-dominated decays must be taken with a grain of salt, since they rely on model-dependent or universality-inspired assumptions of the non-factorizing transverse helicity amplitudes [77].

There is an exception to the power counting for the helicity amplitudes, which arises from a contribution of the electromagnetic dipole operator $Q_{7\gamma}$ to the transverse electroweak penguin amplitude $\alpha_{3,\rm EW}^{p\mp}$ [87], which is *enhanced* by a factor m_b/Λ relative to the longitudinal amplitude (but proportional to the small electromagnetic coupling). This effect manifests itself in branching and polarization fractions of the ρK^* final states and should be seen by comparing precise measurements to theoretical predictions and measurements of the related PP, PV modes πK, ρK, $\pi K^\star$. In particular, polarization measurements would then result in an indirect measurement of the (virtual) photon's polarization in the electromagnetic dipole transition, and hence be sensitive to the possible presence of a dipole operator with opposite chirality of the light quark due to non-standard physics.

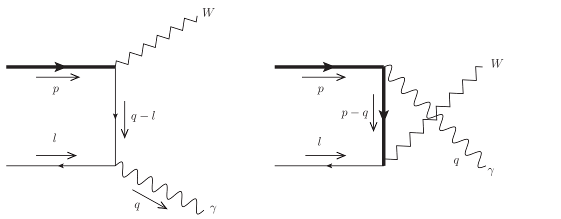

Figure 8: Leading-order diagrams for the $B \to \gamma W^*(\to \ell\nu)$ transition. The left graph shows the leading-power contribution from photon emission from the up antiquark. Emission from the heavy b-quark (right) is power-suppressed. Figure from [88].

4.2. $B \to \gamma\ell\nu$

The decay $B \to \gamma\ell\nu$ is accessible to factorization methods when the energy E_γ of the photon is large compared to Λ. Even though the final state does not contain a hadron, the coupling of the on-shell photon to soft and collinear quarks is non-perturbative and leads to hadronic $\langle 0|\ldots|\bar{B}\rangle$ matrix elements of non-local operators. The two tree diagrams relevant to the process are shown in Figure 8. Photon emission from the heavy quark (right diagram) is power-suppressed relative to the emission from the light quark in the left diagram. The differential branching fraction can be expressed in complete generality in terms of two independent $B \to \gamma$ form factors $F_{V,A}(E_\gamma)$.

Soft-collinear factorization properties of the $B \to \gamma\ell\nu$ amplitude have first been discussed in [89], and in [90, 91] with SCET methods. At leading order in the $1/m_b$ expansion, the two form factors coincide and satisfy a factorization formula

$$F = C \cdot J \otimes \phi_B. \tag{67}$$

In the tree diagram in the left Figure 8 the hard function C is the weak-decay vertex, while the jet function contains the hard-collinear light-quark propagator that connects the W-boson and photon vertices. As in other applications of leading-power factorization, the B-meson LCDA enters only through λ_B and the logarithmic moments (22).

The radiative-leptonic decay is the simplest exclusive decay, and the B-meson LCDA is the only hadronic parameter relevant to this process at leading order in the heavy-quark expansion. It therefore provides a unique way to determine λ_B from (future) data. The sensitivity to λ_B can be seen from the leading-power expression

$$F_{V,A}(E_\gamma) = \frac{Q_u m_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) \tag{68}$$

for the $B \to \gamma$ form factors. The radiative correction factor $R(E_\gamma, \mu)$ and the leading power-suppressed effects have been investigated in [88]. Figure 9 shows the branching fraction with a lower cut on the photon energy as a function of the parameter λ_B and confirms the strong sensitivity. It can be shown that at sub-leading order in Λ/m_b there appears a single unknown "soft" form factor similar to the soft form factor in the $B \to \pi$ transition. However, the difference between F_V and F_A is independent of this form factor and can be given entirely in terms of the B-meson decay constant f_B, at least at tree level [88]. The power-suppressed soft form factor limits the accuracy of the λ_B determination. A numerical estimate has been obtained with the QCD sum-rule method [92].

4.3. Radiative and electroweak penguin decays

I cannot conclude this overview without a brief mentioning of the exclusive radiative and electroweak penguin decays $B \to K^{(*)}\gamma$ and $B \to K^{(*)}\ell^+\ell^-$, respectively. These are primarily driven by the two loop-induced $(\ell\ell)(s\bar{b})$ operators in the weak effective Lagrangian with a non-negligible contribution from the electromagnetic dipole operator and four-quark operators. The hadronic matrix elements of the former two contributions can be parametrized by heavy-to-light form factors, but those of the four-quark operators are non-local. Schematically the transition matrix element contains two terms,

$$\langle K^*\ell\ell|H_{\text{eff}}|\bar{B}\rangle = \sum_i a_i(C^{(\prime)}_{7\gamma,9,10},\ldots)\, F_i^{B\to K^*} + \frac{ie^2}{q^2}\langle\ell\ell|\bar{\ell}\gamma_\mu\ell|0\rangle \int d^4x\, e^{iq\cdot x}\,\langle K^*|T(j^\mu_{\text{em}}(x)H^{\text{had}}_{\text{eff}}(0)|B\rangle, \tag{69}$$

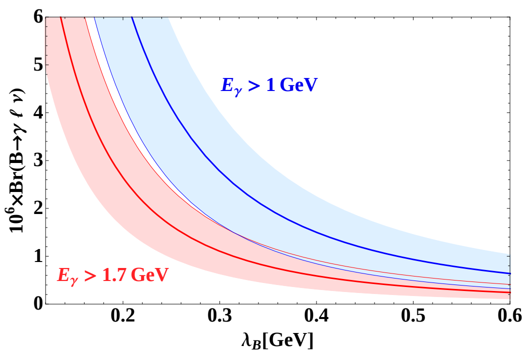

Figure 9: Dependence of the partial branching fractions $\text{Br}(B^- \to \gamma\ell\bar{\nu}, E_\gamma > E_{\text{cut}})$ for $E_{\text{cut}} = 1$ GeV (upper band) and 1.7 GeV (lower band) on λ_B. Figure from [88].

where the first comprises the electroweak penguin and electromagnetic dipole transitions, can be expressed in terms of form factors, and is sensitive to other virtual particles in the loops that generate these transitions. The second is non-local, dominated by QCD effects, including charmonium resonances in the $b \to s(c\bar{c} \to \ell\ell)$ transition.

Soft-collinear factorization is relevant in two ways in the large-recoil region where the kaon energy $E_{K^*} \gg \Lambda$. First, in the first term the QCD form factors can be expressed in terms of only two "soft" form factors $\xi_{\|,\perp}$ through the factorization theorem (3) for form factors. Second, and more importantly, the complicated second term can be calculated and expressed in terms of exactly the same form factors, such that the entire amplitude (69) is expressed as

$$\langle K^*\ell\ell|Q_i|\bar{B}\rangle = C_i\xi + \phi_B \otimes T_i \otimes \phi_{K^*} + O\left(\frac{\Lambda}{m_b}\right) \tag{70}$$

for every operator Q_i in H_{eff} as shown in [93, 94] for the radiative decays $B \to V\gamma$ and in [93] for $B \to V\ell^+\ell^-$. These papers also performed the $O(\alpha_s)$ calculations. Note that NNLO computations for these decay modes are significantly harder than for charmless decays, since at this order three-loop diagrams must be considered for the vertex kernels, and two-loop diagrams for hard-spectator scattering. Only the simpler contributions from the dipole operators have been calculated up to now [95]. The results of [93, 94] were soon generalized to isospin asymmetries [96, 97] and the $B \to \rho\gamma, \rho\ell\ell$ decays [98].

As an example of the uses of factorization I show a plot from [93] for the differential forward-backward asymmetry already discussed above in the context of semi-inclusive hadronic final states. Figure 10 shows the reduction of the theoretical uncertainty at the location of the asymmetry zero q_0^2 due to a reduced sensitivity to the form factors [99, 100] and a sizable shift of q_0^2 from $q_0^2 = 3.4^{+0.6}_{-0.5}\,\text{GeV}^2$ at LO to $q_0^2 = 4.39^{+0.38}_{-0.35}\,\text{GeV}^2$ at NLO. Including an estimate of power corrections to the form factors, $q_0^2 = (4.2 \pm 0.6)\,\text{GeV}^2$ [93] remains a conservative estimate of the asymmetry zero location. A

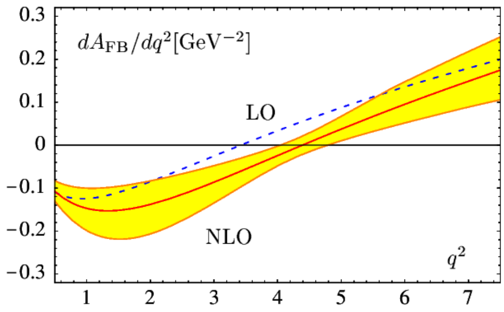

Figure 10: Forward-backward asymmetry $dA_{\rm FB}(B^- \to K^{*-}\ell^+\ell^-)/dq^2$ at next-to-leading order (solid cent line) and leading order (dashed) as function of lepton invariant mass q^2 in GeV. The band reflects all theoretical uncertainties from parameters and scale dependence combined. Figure from [93].

subsequent analysis with updated parameters [98] gives

$$q_0^2[K^{*0}] = 4.36^{+0.33}_{-0.31}\,{\rm GeV}^2 \qquad (71)$$

$$q_0^2[K^{*+}] = 4.15 \pm 0.27\,{\rm GeV}^2 \qquad (72)$$

for the neutral and charged B-meson decay separately, which replaces the value $q_0^2 = 4.39^{+0.38}_{-0.35}$ GeV^2 above.

With the start-up of the LHCb experiment at the horizon and its large anticipated sample of $B \to K^{(*)}\ell\ell$ decays, the fully differential angular distribution including $K^* \to K\pi$ became the subject of investigations [101, 102]. From the angular coefficients one can construct observables with reduced sensitivity to B-meson form factors for all values of q^2 rather than a single value as is the case for the forward-backward asymmetry. This can be done by selecting observables that are proportional to only one of the products $|\xi_\|(q^2)|^2$, $|\xi_\perp(q^2)|^2$, $\xi_\perp(q^2)\xi_\|(q^2)$ of soft form factors, and then taking an appropriate ratio, such that the dependence on ξ cancels out at leading order in α_s. The main theoretical issue is then a reliable estimate of the uncertainty from Λ/m_b corrections, which remains a difficult and partly speculative problem. I refer to [103, 104] on this point and for many references on this very active field of LHCb physics.

5. Summary and Outlook

Factorization of exclusive and semi-inclusive B decays has developed into a mature theory, making a large amount of previously intractable final states accessible to theoretically solid interpretations. The presence of energetic particles or jets implies many similarities with factorization methods applied in high-energy collider physics. A conceptually interesting point is that for exclusive decays the power-suppressed soft interaction $\mathcal{L}^{(1)}_{\xi q} = \bar{q}_s W_c^\dagger i\not{D}_{\perp c}\xi$ + h.c. related to the emission (or rather absorption) of soft quarks $q_c \to g_{hc} + q_s$ appears at leading order in the expansion of the hard scale, whereas in collider physics only the familiar eikonal interaction $\bar\xi i n_- \cdot A_s \frac{\not{n}_+}{2}\xi$ describing soft gluon emission $q_c \to q_c + g_s$ is ever required at leading power. This is because the decaying B meson provides a “source” of soft partons.

Beyond the leading power in the Λ/m_b, the available theoretical tools often cease to be effective. For jet-like, semi-inclusive final states, it appears that soft-collinear factorization can in principle be extended to any order as long as the invariant mass of the inclusive hadronic final state is $O(m_b\Lambda)$. However, in practice already at the first sub-leading order in Λ/m_b a large number of non-perturbative soft functions appears, as was discussed for the case of $\bar{B} \to X_u\ell\bar\nu$. There is presently no method that could compute these functions, since lattice QCD cannot be used for the matrix elements of non-local operators with exactly light-like field separations. For exclusive decays, a theory of power-suppressed effects simply does not exist.

Over the past ten years almost all calculations necessary to push the accuracy of charmless hadronic two-body decays to $O(\alpha_s^2)$ have been performed, except for the two-loop QCD penguin amplitude. Once this is completed, direct CP asymmetries can for the first time be predicted with reliable perturbative uncertainties. In view of the upcoming BELLE II experiment an update of theoretical predictions with NNLO uncertainty and improved hadronic input parameters is timely. There are still many unmeasured, but theoretically predicted final states. Meanwhile, the LHCb experiment yields data on the electroweak penguin decays $\bar{B} \to K^{(*)}\ell\ell$, the precision of which provides or will soon provide a challenge to theory. Extending the theoretical calculations to $O(\alpha_s^2)$ is difficult here, but since the spectator-scattering contribution is presently known only at its first order, the two-loop calculation should be done.

Acknowledgements

This article summarizes work performed within and supported by the DFG Sonderforschungsbereich/Transregio 9 “Computational Theoretical Particle Physics”. I wish to thank many collaborators on this project, in particular Th. Feldmann, T. Huber, S. Jäger, X. Li, J. Rohrwild and D. Yang.

References

[1] E. Eichten, B. R. Hill, An Effective Field Theory for the Calculation of Matrix Elements Involving Heavy Quarks, Phys.Lett. B234 (1990) 511.

[2] B. Grinstein, The Static Quark Effective Theory, Nucl.Phys. B339 (1990) 253–268.
[3] H. Georgi, An Effective Field Theory for Heavy Quarks at Low-energies, Phys.Lett. B240 (1990) 447–450.
[4] M. A. Shifman, M. Voloshin, On Annihilation of Mesons Built from Heavy and Light Quark and $\bar{B}^0 \to B^0$ Oscillations, Sov.J.Nucl.Phys. 45 (1987) 292.
[5] M. A. Shifman, M. Voloshin, On Production of D and D^* Mesons in B Meson Decays, Sov.J.Nucl.Phys. 47 (1988) 511.
[6] I. I. Bigi, M. A. Shifman, N. Uraltsev, A. I. Vainshtein, QCD predictions for lepton spectra in inclusive heavy flavor decays, Phys.Rev.Lett. 71 (1993) 496–499. arXiv:hep-ph/9304225.
[7] N. Isgur, M. B. Wise, Weak Decays of Heavy Mesons in the Static Quark Approximation, Phys.Lett. B232 (1989) 113–117.
[8] N. Isgur, M. B. Wise, Weak Transition Form-factors between Heavy Mesons, Phys.Lett. B237 (1990) 527–530.
[9] M. Beneke, G. Buchalla, M. Neubert, C. T. Sachrajda, QCD factorization for $B \to \pi\pi$ decays: Strong phases and CP violation in the heavy quark limit, Phys.Rev.Lett. 83 (1999) 1914–1917. arXiv:hep-ph/9905312.
[10] M. Beneke, G. Buchalla, M. Neubert, C. T. Sachrajda, QCD factorization for exclusive, nonleptonic B meson decays: General arguments and the case of heavy light final states, Nucl.Phys. B591 (2000) 313–418. arXiv:hep-ph/0006124.
[11] M. Beneke, G. Buchalla, M. Neubert, C. T. Sachrajda, QCD factorization in $B \to \pi K, \pi\pi$ decays and extraction of Wolfenstein parameters, Nucl.Phys. B606 (2001) 245–321. arXiv:hep-ph/0104110.
[12] J. D. Bjorken, Topics in B Physics, Nucl.Phys.Proc.Suppl. 11 (1989) 325–341.
[13] H. D. Politzer, M. B. Wise, Perturbative corrections to factorization in anti-B decay, Phys.Lett. B257 (1991) 399–402.
[14] G. P. Lepage, S. J. Brodsky, Exclusive Processes in Perturbative Quantum Chromodynamics, Phys.Rev. D22 (1980) 2157.
[15] A. Efremov, A. Radyushkin, Factorization and Asymptotical Behavior of Pion Form-Factor in QCD, Phys.Lett. B94 (1980) 245–250.
[16] C. W. Bauer, S. Fleming, M. E. Luke, Summing Sudakov logarithms in $B \to X_s\gamma$ in effective field theory, Phys. Rev. D63 (2001) 014006. arXiv:hep-ph/0005275.
[17] C. W. Bauer, S. Fleming, D. Pirjol, I. W. Stewart, An effective field theory for collinear and soft gluons: Heavy to light decays, Phys. Rev. D63 (2001) 114020. arXiv:hep-ph/0011336.
[18] C. W. Bauer, D. Pirjol, I. W. Stewart, Soft-collinear factorization in effective field theory, Phys. Rev. D65 (2002) 054022. arXiv:hep-ph/0109045.
[19] M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, Soft collinear effective theory and heavy to light currents beyond leading power, Nucl.Phys. B643 (2002) 431–476. arXiv:hep-ph/0206152.
[20] M. Beneke, T. Feldmann, Multipole expanded soft collinear effective theory with non-abelian gauge symmetry, Phys.Lett. B553 (2003) 267–276. arXiv:hep-ph/0211358.
[21] C. W. Bauer, D. Pirjol, I. W. Stewart, A Proof of factorization for $B \to D\pi$, Phys.Rev.Lett. 87 (2001) 201806. arXiv:hep-ph/0107002.
[22] A. Szczepaniak, E. M. Henley, S. J. Brodsky, Perturbative QCD Effects in Heavy Meson Decays, Phys.Lett. B243 (1990) 287–292.
[23] J. Charles, A. Le Yaouanc, L. Oliver, O. Pene, J. Raynal, Heavy to light form-factors in the heavy mass to large energy limit of QCD, Phys.Rev. D60 (1999) 014001. arXiv:hep-ph/9812358.
[24] M. Beneke, T. Feldmann, Symmetry breaking corrections to heavy to light B meson form-factors at large recoil, Nucl.Phys. B592 (2001) 3–34. arXiv:hep-ph/0008255.
[25] J. Chay, C. Kim, Collinear effective theory at subleading order and its application to heavy - light currents, Phys.Rev. D65 (2002) 114016. arXiv:hep-ph/0201197.
[26] C. W. Bauer, D. Pirjol, I. W. Stewart, Factorization and endpoint singularities in heavy to light decays, Phys.Rev. D67 (2003) 071502. arXiv:hep-ph/0211069.
[27] M. Beneke, T. Feldmann, Factorization of heavy to light form-factors in soft collinear effective theory, Nucl.Phys. B685 (2004) 249–296. arXiv:hep-ph/0311335.
[28] B. O. Lange, M. Neubert, Factorization and the soft overlap contribution to heavy to light form-factors, Nucl.Phys. B690 (2004) 249–278. arXiv:hep-ph/0311345.
[29] A. Grozin, M. Neubert, Asymptotics of heavy meson form-factors, Phys.Rev. D55 (1997) 272–290. arXiv:hep-ph/9607366.
[30] M. Beneke, Soft-collinear effective theory, Lectures given at the Helmholtz International School on Heavy Quark Physics, June 6-16, 2005, Dubna, Russia, http://theor.jinr.ru/ hq2005/PDF.html.
[31] M. Beneke, Y. Kiyo, D. Yang, Loop corrections to subleading heavy quark currents in SCET, Nucl.Phys. B692 (2004) 232–248. arXiv:hep-ph/0402241.
[32] T. Becher, R. J. Hill, Loop corrections to heavy-to-light form-factors and evanescent operators in SCET, JHEP 0410 (2004) 055. arXiv:hep-ph/0408344.
[33] R. Hill, T. Becher, S. J. Lee, M. Neubert, Sudakov resummation for subleading SCET currents and heavy-to-light form-factors, JHEP 0407 (2004) 081. arXiv:hep-ph/0404217.
[34] M. Beneke, D. Yang, Heavy-to-light B meson form-factors at large recoil energy: Spectator-scattering corrections, Nucl.Phys. B736 (2006) 34–81. arXiv:hep-ph/0508250.
[35] J. Chay, C. Kim, Factorization of B decays into two light mesons in soft collinear effective theory, Phys.Rev. D68 (2003) 071502. arXiv:hep-ph/0301055.
[36] J. Chay, C. Kim, Nonleptonic B decays into two light mesons in soft collinear effective theory, Nucl.Phys. B680 (2004) 302–338. arXiv:hep-ph/0301262.
[37] C. W. Bauer, D. Pirjol, I. Z. Rothstein, I. W. Stewart, $B \to M_1M_2$: Factorization, charming penguins, strong phases, and polarization, Phys.Rev. D70 (2004) 054015. arXiv:hep-ph/0401188.
[38] M. Beneke, S. Jager, Spectator scattering at NLO in non-leptonic B decays: Tree amplitudes, Nucl.Phys. B751 (2006) 160–185. arXiv:hep-ph/0512351.
[39] B. O. Lange, M. Neubert, Renormalization group evolution of the B meson light cone distribution amplitude, Phys.Rev.Lett. 91 (2003) 102001. arXiv:hep-ph/0303082.
[40] Y. Keum, H.-N. Li, A. Sanda, Penguin enhancement and B to K pi decays in perturbative QCD, Phys.Rev. D63 (2001) 054008. arXiv:hep-ph/0004173.
[41] C.-D. Lu, K. Ukai, M.-Z. Yang, Branching ratio and CP violation of $B \to \pi\pi$ decays in perturbative QCD approach, Phys.Rev. D63 (2001) 074009. arXiv:hep-ph/0004213.
[42] H.-n. Li, S. Mishima, Possible resolution of the $B \to \pi\pi, \pi K$ puzzles, Phys.Rev. D83 (2011) 034023. arXiv:0901.1272.
[43] M. Ciuchini, E. Franco, G. Martinelli, M. Pierini, L. Silvestrini, Charming penguins strike back, Phys.Lett. B515 (2001) 33–41. arXiv:hep-ph/0104126.
[44] M. Beneke, G. Buchalla, M. Neubert, C. Sachrajda, Comment on $B \to M_1M_2$: Factorization, charming penguins, strong phases, and polarization', Phys.Rev. D72 (2005) 098501. arXiv:hep-ph/0411171.
[45] M. Beneke, G. Buchalla, M. Neubert, C. Sachrajda, Penguins with Charm and Quark-Hadron Duality, Eur.Phys.J. C61 (2009) 439–449. arXiv:0902.4446.

[46] Z.-z. Song, K.-T. Chao, Problems of QCD factorization in exclusive decays of B meson to charmonium, Phys.Lett. B568 (2003) 127–134. arXiv:hep-ph/0206253.
[47] M. Beneke, L. Vernazza, $B \to \chi_{cJ} K$ decays revisited, Nucl.Phys. B811 (2009) 155–181. arXiv:0810.3575.
[48] G. P. Korchemsky, G. F. Sterman, Infrared factorization in inclusive B meson decays, Phys.Lett. B340 (1994) 96–108. arXiv:hep-ph/9407344.
[49] K. S. Lee, I. W. Stewart, Factorization for power corrections to $B \to X_s\gamma$ and $B \to X_u \ell\nu$, Nucl.Phys. B721 (2005) 325–406. arXiv:hep-ph/0409045.
[50] S. W. Bosch, M. Neubert, G. Paz, Subleading shape functions in inclusive B decays, JHEP 0411 (2004) 073. arXiv:hep-ph/0409115.
[51] M. Beneke, F. Campanario, T. Mannel, B. Pecjak, Power corrections to $\bar{B} \to X_u \ell\bar{\nu}(X_s\gamma)$ decay spectra in the 'shape-function' region, JHEP 0506 (2005) 071. arXiv:hep-ph/0411395.
[52] H. Asatrian, C. Greub, B. Pecjak, NNLO corrections to $\bar{B} \to X_u \ell\bar{\nu}$ in the shape-function region, Phys.Rev. D78 (2008) 114028. arXiv:0810.0987.
[53] R. Bonciani, A. Ferroglia, Two-Loop QCD Corrections to the Heavy-to-Light Quark Decay, JHEP 0811 (2008) 065. arXiv:0809.4687.
[54] M. Beneke, T. Huber, X.-Q. Li, Two-loop QCD correction to differential semi-leptonic $b \to u$ decays in the shape-function region, Nucl.Phys. B811 (2009) 77–97. arXiv:0810.1230.
[55] G. Bell, NNLO corrections to inclusive semileptonic B decays in the shape-function region, Nucl.Phys. B812 (2009) 264–289. arXiv:0810.5695.
[56] G. Bell, M. Beneke, T. Huber, X.-Q. Li, Heavy-to-light currents at NNLO in SCET and semi-inclusive $\bar{B} \to X_s \ell^+\ell^-$ decay, Nucl.Phys. B843 (2011) 143–176. arXiv:1007.3758.
[57] P. Ball, R. Zwicky, New results on $B \to \pi, K, \eta$ decay form factors from light-cone sum rules, Phys.Rev. D71 (2005) 014015. arXiv:hep-ph/0406232.
[58] P. Ball, R. Zwicky, $B_{d,s} \to \rho, \omega, K^*, \phi$ decay form-factors from light-cone sum rules revisited, Phys.Rev. D71 (2005) 014029. arXiv:hep-ph/0412079.
[59] K. S. Lee, Z. Ligeti, I. W. Stewart, F. J. Tackmann, Universality and m_X cut effects in $B \to X_s \ell^+\ell^-$, Phys.Rev. D74 (2006) 011501. arXiv:hep-ph/0512191.
[60] K. S. Lee, F. J. Tackmann, Nonperturbative m_X cut effects in $B \to X_s \ell^+\ell^-$ observables, Phys.Rev. D79 (2009) 114021. arXiv:0812.0001.
[61] T. Huber, T. Hurth, E. Lunghi, Logarithmically Enhanced Corrections to the Decay Rate and Forward Backward Asymmetry in $\bar{B} \to X_s \ell^+\ell^-$, Nucl.Phys. B802 (2008) 40–62. arXiv:0712.3009.
[62] N. Kivel, Radiative corrections to hard spectator scattering in $B \to \pi\pi$ decays, JHEP 0705 (2007) 019. arXiv:hep-ph/0608291.
[63] V. Pilipp, Hard spectator interactions in $B \to \pi\pi$ at order α_s^2, Nucl.Phys. B794 (2008) 154–188. arXiv:0709.3214.
[64] G. Bell, NNLO vertex corrections in charmless hadronic B decays: Imaginary part, Nucl.Phys. B795 (2008) 1–26. arXiv:0705.3127.
[65] G. Bell, NNLO vertex corrections in charmless hadronic B decays: Real part, Nucl.Phys. B822 (2009) 172–200. arXiv:0902.1915.
[66] M. Beneke, T. Huber, X.-Q. Li, NNLO vertex corrections to non-leptonic B decays: Tree amplitudes, Nucl.Phys. B832 (2010) 109–151. arXiv:0911.3655.
[67] G. Passarino, M. Veltman, One Loop Corrections for e+ e- Annihilation Into mu+ mu- in the Weinberg Model, Nucl.Phys. B160 (1979) 151.
[68] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, Int.J.Mod.Phys. A15 (2000) 5087–5159. arXiv:hep-ph/0102033.
[69] F. Tkachov, A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions, Phys.Lett. B100 (1981) 65–68.
[70] K. Chetyrkin, F. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, Nucl.Phys. B192 (1981) 159–204.
[71] M. Beneke, M. Neubert, QCD factorization for $B \to PP$ and $B \to PV$ decays, Nucl.Phys. B675 (2003) 333–415. arXiv:hep-ph/0308039.
[72] G. Bell, V. Pilipp, $B^- \to \pi^-\pi^0/\rho^-\rho^0$ to NNLO in QCD factorization, Phys.Rev. D80 (2009) 054024. arXiv:0907.1016.
[73] Y. Amhis, et al., Averages of b-hadron, c-hadron, and τ-lepton properties as of summer 2014, arXiv:1412.7515.
[74] M. Beneke, S. Jager, Spectator scattering at NLO in non-leptonic B decays: Leading penguin amplitudes, Nucl.Phys. B768 (2007) 51–84. arXiv:hep-ph/0610322.
[75] G. Bell, T. Huber, Master integrals for the two-loop penguin contribution in non-leptonic B-decays, arXiv:1410.2804.
[76] A. Bevan, et al., The Physics of the B Factories, Eur.Phys.J. C74 (11) (2014) 3026. arXiv:1406.6311.
[77] M. Beneke, J. Rohrer, D. Yang, Branching fractions, polarisation and asymmetries of $B \to VV$ decays, Nucl.Phys. B774 (2007) 64–101. arXiv:hep-ph/0612290.
[78] H.-Y. Cheng, C.-K. Chua, K.-C. Yang, Charmless B decays to a scalar meson and a vector meson, Phys.Rev. D77 (2008) 014034. arXiv:0705.3079.
[79] H.-Y. Cheng, K.-C. Yang, Hadronic charmless B decays $B \to AP$, Phys.Rev. D76 (2007) 114020. arXiv:0709.0137.
[80] H.-Y. Cheng, K.-C. Yang, Branching Ratios and Polarization in B to VV, VA, AA Decays, Phys.Rev. D78 (2008) 094001. arXiv:0805.0329.
[81] H.-Y. Cheng, K.-C. Yang, Charmless Hadronic B Decays into a Tensor Meson, Phys.Rev. D83 (2011) 034001. arXiv:1010.3309.
[82] M. Beneke, M. Neubert, Flavor singlet B decay amplitudes in QCD factorization, Nucl.Phys. B651 (2003) 225–248. arXiv:hep-ph/0210085.
[83] A. L. Kagan, Polarization in B to VV decays, Phys.Lett. B601 (2004) 151–163. arXiv:hep-ph/0405134.
[84] M. Beneke, Corrections to $\sin(2\beta)$ from CP asymmetries in $B^0 \to (\pi^0, \rho^0, \eta, \eta', \omega, \phi)K_S$ decays, Phys.Lett. B620 (2005) 143–150. arXiv:hep-ph/0505075.
[85] H.-Y. Cheng, C.-K. Chua, A. Soni, Effects of final-state interactions on mixing-induced CP violation in penguin-dominated B decays, Phys.Rev. D72 (2005) 014006. arXiv:hep-ph/0502235.
[86] M. Beneke, Hadronic B decays, Nucl.Phys.Proc.Suppl. 170 (2007) 57–62. arXiv:hep-ph/0612353.
[87] M. Beneke, J. Rohrer, D. Yang, Enhanced electroweak penguin amplitude in $B \to VV$ decays, Phys.Rev.Lett. 96 (2006) 141801. arXiv:hep-ph/0512258.
[88] M. Beneke, J. Rohrwild, B meson distribution amplitude from $B \to \gamma\ell\nu$, Eur.Phys.J. C71 (2011) 1818. arXiv:1110.3228.
[89] G. P. Korchemsky, D. Pirjol, T.-M. Yan, Radiative leptonic decays of B mesons in QCD, Phys.Rev. D61 (2000) 114510. arXiv:hep-ph/9911427.
[90] E. Lunghi, D. Pirjol, D. Wyler, Factorization in leptonic radiative $B \to \gamma e\nu$ decays, Nucl.Phys. B649 (2003) 349–364. arXiv:hep-ph/0210091.
[91] S. Bosch, R. Hill, B. Lange, M. Neubert, Factorization and Sudakov resummation in leptonic radiative B decay, Phys.Rev.

D67 (2003) 094014. arXiv:hep-ph/0301123.

[92] V. Braun, A. Khodjamirian, Soft contribution to $B \to \gamma \ell \nu_\ell$ and the B-meson distribution amplitude, Phys.Lett. B718 (2013) 1014–1019. arXiv:1210.4453.

[93] M. Beneke, T. Feldmann, D. Seidel, Systematic approach to exclusive $B \to V\ell^+\ell^-, V\gamma$ decays, Nucl.Phys. B612 (2001) 25–58. arXiv:hep-ph/0106067.

[94] S. W. Bosch, G. Buchalla, The radiative decays $B \to V\gamma$ at next-to-leading order in QCD, Nucl.Phys. B621 (2002) 459–478. arXiv:hep-ph/0106081.

[95] A. Ali, B. D. Pecjak, C. Greub, $B \to$ V γ Decays at NNLO in SCET, Eur.Phys.J. C55 (2008) 577–595. arXiv:0709.4422.

[96] A. L. Kagan, M. Neubert, Isospin breaking in $B \to K^*\gamma$ decays, Phys.Lett. B539 (2002) 227–234. arXiv:hep-ph/0110078.

[97] T. Feldmann, J. Matias, Forward backward and isospin asymmetry for $B \to K^*\ell^+\ell^-$ decay in the standard model and in supersymmetry, JHEP 0301 (2003) 074. arXiv:hep-ph/0212158.

[98] M. Beneke, T. Feldmann, D. Seidel, Exclusive radiative and electroweak $b \to d$ and $b \to s$ penguin decays at NLO, Eur.Phys.J. C41 (2005) 173–188. arXiv:hep-ph/0412400.

[99] G. Burdman, Short distance coefficients and the vanishing of the lepton asymmetry in $B \to V\ell^+\ell^-$, Phys.Rev. D57 (1998) 4254–4257. arXiv:hep-ph/9710550.

[100] A. Ali, P. Ball, L. Handoko, G. Hiller, A Comparative study of the decays $B \to (K, K^*)\ell^+\ell^-$ in standard model and supersymmetric theories, Phys.Rev. D61 (2000) 074024. arXiv:hep-ph/9910221.

[101] U. Egede, T. Hurth, J. Matias, M. Ramon, W. Reece, New observables in the decay mode $\bar{B}_d \to \bar{K}^{*0}\ell^+\ell^-$, JHEP 0811 (2008) 032. arXiv:0807.2589.

[102] W. Altmannshofer, P. Ball, A. Bharucha, A. J. Buras, D. M. Straub, et al., Symmetries and Asymmetries of $B \to K^*\mu^+\mu^-$ Decays in the Standard Model and Beyond, JHEP 0901 (2009) 019. arXiv:0811.1214.

[103] S. Jager, J. Martin Camalich, On $B \to V\ell\ell$ at small dilepton invariant mass, power corrections, and new physics, JHEP 1305 (2013) 043. arXiv:1212.2263.

[104] S. Descotes-Genon, L. Hofer, J. Matias, J. Virto, On the impact of power corrections in the prediction of $B \to K^*\mu^+\mu^-$ observables, JHEP 1412 (2014) 125. arXiv:1407.8526.